Example1. 2015:-

Show that the transformation defined by

$$Q = \sqrt{2q}\, e^{t} \cos p, \quad P = \sqrt{2q}\, e^{-t} \sin p$$

is canonical.

Solution.

We have to show that $\delta F = p\delta q - P\delta Q$ is an exact differential.

As $\delta Q = \frac{\partial Q}{\partial q}\delta q + \frac{\partial Q}{\partial p}\delta p + \frac{\partial Q}{\partial t}\delta t \to 0$

so $\delta Q = \frac{\partial Q}{\partial q}\delta q + \frac{\partial Q}{\partial p}\delta p$ ——— (1)

Since $Q = \sqrt{2q}\, e^{t} \cos p = \sqrt{2}\sqrt{q}\, e^{t} \cos p$

Therefore, $\frac{\partial Q}{\partial q} = \sqrt{2} \cdot \frac{1}{2} (q)^{-1/2} e^{t} \cos p$

$\Rightarrow \frac{\partial Q}{\partial q} = \frac{1}{\sqrt{2q}} e^{t} \cos p$ ——— (2)

and $\frac{\partial Q}{\partial p} = -\sqrt{2q}\, e^{t} \sin p$ ——— (3)

Subsituting (2) and (3) in (1), we get,

$$\delta Q = \left(\frac{1}{\sqrt{2q}} e^{t} \cos p\right)\delta q + \left(-\sqrt{2q}\, e^{t} \sin p\right)\delta p$$

Thus $\delta F = p\delta q - \sqrt{2q}\, e^{-t} \sin p \left(\frac{1}{\sqrt{2q}} e^{t} \cos p\, \delta q - \sqrt{2q}\right.$

$= p\delta q - \sin p \cos p\, \delta q + (2q) \sin^2 p\, \delta p$

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$\delta F = (p - \sin p \cos p)\delta q + (2q \sin^2 p)\delta p$

$\delta F = M(q,p)\delta q + N(q,p)\delta p$ ——— (4)

where $M(q,p) = p - \sin p \cos p$

$N(q,p) = 2q \sin^2 p$

The R.H.S. of equation (4) is exact differential a function $F(q,p)$ if $\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$

that

$\frac{\partial M}{\partial p} = 1 - \frac{\partial}{\partial p}(\sin p \cos p) = 1 - \frac{1}{2}\frac{\partial}{\partial p}(2 \sin p \cos p)$

$= 1 - \frac{1}{2}\frac{\partial}{\partial p}(\sin 2p) = 1 - \frac{1}{2}(\cos 2p) \cdot 2$

$\frac{\partial M}{\partial p} = 1 - \cos 2p = 2\sin^2 p$ ——— (5)

$\frac{\partial N}{\partial q} = 2\sin^2 p$ ——— (6)

Thus equation (5) and eq (6) yeild

$\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$.

Hence R.H.S. of (4) is an exact differential.

Therefore, the given transformation is canonical.

To obtain generator function we proceed as follow:

$M(q,p) = \frac{\partial F}{\partial q}$, $N(q,p) = \frac{\partial F}{\partial p}$

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We know that $\vec{F}\cdot d\vec{s} = dq \Rightarrow dq = F_1 dx + F_2 dy$

Thus $dF = M(q,p)dq + N(q,p)dp$

$= (M\hat{i} + N\hat{j})\cdot(dq\hat{i} + dp\hat{j})$

$\Rightarrow dF = \frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial p}dp$

Common terms ...

Example 2:

Show that the transformation $Q = \frac{1}{2}(p^2+q^2)$; $P = -\tan^{-1}(q/p)$ is canonical. Also find the form F_1 of the generating function.

Solution.

To show that the given transformation is canonical we require that $p\delta q - P\delta Q$ be the exact differential of some function F.

i.e. $\delta F = p\delta q - P\delta Q$ ——— (1)

Now we find δQ with the help of given

$\delta Q = \frac{\partial Q}{\partial q}\delta q + \frac{\partial Q}{\partial p}\delta p$ ——— (2)

Since $Q = \frac{1}{2}(p^2+q^2)$

Therefore, $\frac{\partial Q}{\partial q} = q$ and $\frac{\partial Q}{\partial p} = p$

(2) $\Rightarrow \delta Q = q\delta q + p\delta p$

Thus equation (1) becomes,

$\delta F = p\delta q - P(q\delta q + p\delta p)$

$= p\delta q + \tan^{-1}(q/p)(q\delta q + p\delta p)$ using gi... value of

$= p\delta q + \tan^{-1}(\frac{q}{p})\, q\delta q + \tan^{-1}(q/p)\, p\delta p$

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$= [p + q\tan^{-1}(q/p)]\delta q + [p\tan^{-1}(q/p)]\delta p$

$= M(q,p)\,\delta q + N(q,p)\,\delta p \longrightarrow (3)$

where $M(q,p) = p + q\tan^{-1}(q/p)$

$N(q,p) = p\tan^{-1}(q/p)$

R.H.S of equation (3) is exact differential of function $F(q,p)$ if $\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$.

Now that $\frac{\partial M}{\partial p} = 1 + q\left(\frac{p^2}{p^2+q^2}\right)\left(-\frac{q}{p^2}\right) = 1 - \frac{q^2}{p^2+q^2}$

$\Rightarrow \frac{\partial M}{\partial p} = \frac{p^2+q^2-q^2}{p^2+q^2} = \frac{p^2}{p^2+q^2} \longrightarrow (4)$

$\frac{\partial N}{\partial q} = p\left(\frac{p^2}{p^2+q^2}\right)\left(\frac{1}{p}\right) = \frac{p^2}{p^2+q^2} \longrightarrow (5)$

Thus from (4) and (5), we can write,

$$\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$$

Hence the R.H.S of (3) is an exact diff. of some function F. Therefore the given transformation is Canonical.

Now we find the generating function F as follows,

$\Rightarrow dF = [p + q\tan^{-1}(q/p)]\,dq + [p\tan^{-1}(q/p)]\,dp$

$F \cdot d\vec{x} = d\phi$

Integrating, we get.

$F = \int [p + q\tan^{-1}(q/p)]\,dq + \int [p\tan^{-1}(q/p)]\,dp \longrightarrow (6)$

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Note that

$$\int [p + q\tan^{-1}(q/p)]\,dq = pq + \frac{q^2}{2}\tan^{-1}(q/p) - \int \frac{q^2}{2}\cdot\frac{p^2}{p^2+q^2}(\cdots$$

$$= pq + \frac{q^2}{2}\tan^{-1}(q/p) - \int \frac{pq^2}{2(p^2+q^2)}\,dq$$

$$= pq + \frac{q^2}{2}\tan^{-1}\left(\frac{q}{p}\right) - \frac{p}{2}\int \frac{(p^2+q^2-p^2)}{p^2+q^2}\,dq$$

$$= pq + \frac{q^2}{2}\tan^{-1}\left(\frac{q}{p}\right) - \frac{p}{2}\int \left(1 - \frac{p^2}{p^2+q^2}\right)dq$$

$$= pq + \frac{q^2}{2}\tan^{-1}(q/p) - \frac{p}{2}\left[\int dq - p\int \frac{p}{p^2+q^2}\,dq\right]$$

$$= pq + \frac{q^2}{2}\tan^{-1}(q/p) - \frac{1}{2}pq + \frac{1}{2}p^2\tan^{-1}(q/p)$$

$$= \frac{1}{2}pq + \frac{1}{2}(p^2+q^2)\tan^{-1}(q/p) \qquad \text{——— (7)}$$

And

$$\int [p\tan^{-1}(q/p)]\,dp = \frac{p^2}{2}\tan^{-1}\left(\frac{q}{p}\right) - \int \frac{p^2}{2}\cdot\frac{p^2}{p^2+q^2}(-q/p$$

$$= p/2\,\tan^{-1}(q/p) + \frac{1}{2}\int \frac{p^2 q}{p^2+q^2}\,dp$$

$$= p/2\,\tan^{-1}(q/p) + \frac{q}{2}\int \frac{p^2+q^2-q^2}{p^2+q^2}\,dp.$$

$$= p/2\,\tan^{-1}(q/p) + \frac{q}{2}\int \left(1 - \frac{q^2}{p^2+q^2}\right)dp$$

$$= \frac{p^2}{2}\tan^{-1}(q/p) + \frac{q}{2}\left[\int dp + q\int \frac{-q\,dp}{p^2+q^2}\right]$$

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$= \frac{p^2}{2}\tan^{-1}(q/p) + \frac{1}{2}pq + \frac{1}{2}q^2\tan^{-1}(q/p)$

$= \frac{1}{2}pq + \frac{1}{2}(p^2+q^2)\tan^{-1}(q/p)$ ——— (8)

om (7) and (8), we put common value in (6) as

llows

$= \frac{1}{2}pq + \frac{1}{2}(p^2+q^2)\tan^{-1}(q/p)$ ——— (9)

hich is our required generating function.

order to find the $F_1(t,q,Q)$ of the g.function, need to introduce Q in (9) and eliminate p.

e that

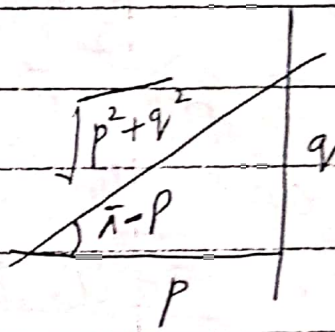
$= \frac{1}{2}(p^2+q^2)$; $P = -\tan^{-1}(q/p)$

$Q = p^2+q^2$; $\Rightarrow \tan(-P) = q/p$

$Q - q^2 = p^2$; $\Rightarrow \tan(\pi - P) = q/p$ ——→ (11)

$= \sqrt{2Q-q^2}$ ——→ (10)

om Fig. we can write.



$\sin(\pi - P) = \frac{q}{\sqrt{q^2+p^2}}$

$\sin P = \frac{q}{\sqrt{p^2+q^2}}$

$P = \sin^{-1}\left(\frac{q}{\sqrt{2Q}}\right) \Rightarrow -P = -\sin^{-1}\left(\frac{q}{\sqrt{2Q}}\right)$

s $-P = \tan^{-1}(q/p)$.

$\tan^{-1}(q/p) = -\sin^{-1}\left(q/\sqrt{2Q}\right)$ ——— (12)

Thus equation (9) becomes

$F = \frac{1}{2}q\sqrt{2Q-q^2} - Q\sin^{-1}\left(\frac{q}{\sqrt{2Q}}\right) = F_1(t,q,Q)$.

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OR ⑨ $\Rightarrow F = \frac{1}{2} q\sqrt{2Q - q^2} + Q \tan^{-1}\left(\frac{q}{\sqrt{2Q-q^2}}\right) = F_1(t, q$

which is our required form F_1 of generating function.

Now we find $F_2(t, q, P)$, $F_3(t, p, Q)$ and $F_4(t, p$ as follows:

For F_2 we have to introduce P and eliminate p in ⑨.

Note that $P = -\tan^{-1}(q/p) \Rightarrow \tan(-P) = q/p$

$\Rightarrow -\tan(P) = q/p \Rightarrow p = -q \cot P$

⑨ $\Rightarrow F = -\frac{1}{2} q^2 \cot P + \frac{1}{2}[q^2 \cot^2 P + q^2] \tan^{-1}\left(\frac{-q}{q \cot P}\right)$

$= -\frac{1}{2} q^2 \cot P + \frac{1}{2}(q^2 \operatorname{cosec}^2 P) \tan^{-1}(-\tan P)$.

$= -\frac{1}{2} q^2 \cot P - \frac{1}{2} P q^2 \operatorname{cosec}^2 P = F_2(t, q, P)$.

To find F_3 we need to introduce Q in ⑨ and eliminate q.

Note that $Q = \frac{1}{2}(p^2 + q^2) \Rightarrow q = \sqrt{2Q - p^2}$

⑨ $\Rightarrow F = \frac{1}{2} P\sqrt{2Q - p^2} + Q \tan^{-1}\left(\frac{\sqrt{2Q-p^2}}{p}\right) = F_3(t, p,$

To find F_4 we need to introduce P in ⑨ and eliminate q.

Note that $P = -\tan^{-1}(q/p) \Rightarrow \tan(-P) = q/p$

$\Rightarrow q = -p \tan P$.

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$$F = -\frac{1}{2}P^2\tan Q + \frac{1}{2}(P^2 + P^2\tan^2 Q)\tan^{-1}\left(\frac{-P\tan Q}{P}\right)$$

$$= -\frac{1}{2}P^2\tan Q + \frac{1}{2}P^2\sec^2 Q\,(-Q)$$

$$= -\frac{1}{2}P^2\tan Q - \frac{1}{2}P^2 Q\sec^2 Q = F_4(t, P, Q).$$

Exercise

ove that the following transformations
e canonical and find the generator
transformation.

$P = \frac{1}{2}(p^2+q^2)$, $Q = \tan^{-1}(q/p)$. 2012 2014

olution.

We show that $pdq - PdQ$ is the exact differential of some function F.

i.e. $dF = pdq - PdQ$ ——— (1)

Now we find dQ from given $Q = \tan^{-1}(q/$

$$dQ = \frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp \quad \text{——— (2)}$$

$$\frac{\partial Q}{\partial q} = \frac{p^2}{p^2+q^2}\left(\frac{1}{p}\right) = \frac{p}{p^2+q^2}$$

$$\text{nd } \frac{\partial Q}{\partial p} = \frac{p^2}{p^2+q^2}\left(\frac{-q}{p^2}\right) = \frac{-q}{p^2+q^2}$$

$$\text{(2)}\quad dQ = \frac{pdq - qdp}{p^2+q^2}$$

$$\text{(1)} \Rightarrow dF = pdq - \frac{1}{2}(p^2+q^2)\left(\frac{pdq - qdp}{p^2+q^2}\right)$$

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$$\Rightarrow dF = pdq - \frac{1}{2}(pdq - qdp)$$

$$= pdq - \frac{1}{2}pdq + \frac{1}{2}qdp$$

$$\Rightarrow dF = \frac{1}{2}pdq + \frac{1}{2}qdp \longrightarrow (3)$$

$$= M(q,p)dq + N(q,p)dp$$

where $M = \frac{1}{2}p$ and $N = \frac{1}{2}q$

The R.H.S of (3) is exact differential if

$$\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$$

Note that $\frac{\partial M}{\partial p} = \frac{1}{2} = \partial N/\partial q$

Hence the R.H.S of (3) is exact differen
of some function F, so the given transfor
is canonical.

Now we find the generating function

(3) $\Rightarrow dF = \frac{1}{2}pdq + \frac{1}{2}qdp = \frac{1}{2}d(pq)$

Integrating, we get,

$$F = \frac{1}{2}pq$$

which is our desired generating functio

(2) April 7 $p = m\omega q \cot Q$ and $P = \frac{m\omega q^2}{2\sin^2 Q}$

5/2018 Solution.

Given that

$$p = m\omega q \cot Q \quad ; P = \frac{m\omega q^2}{2\sin^2 Q}$$

Therefore,

$$\cot Q = \frac{p}{m\omega q}$$

Taking differential, we get

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$\cosec^2 Q\, dQ = \frac{1}{m\omega}\left(-\frac{p}{q^2}dq + \frac{dp}{q}\right)$

$\Rightarrow dQ = \frac{1}{m\omega}\left(\frac{p}{q^2}dq - \frac{dp}{q}\right)\sin^2 Q$

thus $dF = pdq - \frac{m\omega q^2}{2\sin^2 Q}\cdot\frac{1}{m\omega}\left(\frac{p}{q^2}dq - \frac{dp}{q}\right)\sin Q$

$= pdq - \frac{1}{2}q^2\left(\frac{p}{q^2}dq - dp/q\right)$

$= pdq - \frac{1}{2}pdq + \frac{1}{2}q\,dp$

$= \frac{1}{2}pdq + \frac{1}{2}q\,dp$ (proceed)

which is same as above question.

3) $P = q\cot p$ and $Q = \ln\left(\frac{1}{q}\sin p\right)$.

Solution.

Given that $Q = \ln\left(\frac{1}{q}\sin p\right)$

Therefore, $dQ = \frac{q}{\sin p}\left(\frac{-1}{q^2}\sin p\,dq + \frac{1}{q}\cos p\,dp\right)$

$= \frac{-1}{q}dq + \cot p\,dp$

thus $dF = pdq - q\cot p\left(\frac{-1}{q}dq + \cot p\,dp\right)$

$= pdq + \cot p\,dq - q\cot^2 p\,dp$

$\Rightarrow dF = (p + \cot p)dq - q\cot^2 p\,dp$ ——— (1)

$= M(q,p)dq + N(q,p)dp$

where $M = p + \cot p$ and $N = -q\cot^2 p$

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the R.H.S. of ① is exact differential if $\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$

Note that $\frac{\partial M}{\partial p} = 1 - \operatorname{cosec}^2 p$ $\quad \because \operatorname{cosec}^2\theta = 1+$

$= -\cot^2 p$

and $\frac{\partial N}{\partial q} = -\cot^2 p$

$\Rightarrow \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$

This shows that the R.H.S of ① is exa differential of F, so the transformation is canon

Now we find generating function F a

① $dF = (p+\cot p)dq - q\cot^2 p\,dp$ —— ②

Integrating, we have,

$F = \int (p+\cot p)dq - \int q\cot^2 p\,dp$ —— ③

Note that $\int (p+\cot p)dq = (p+\cot p)q$ —— ④

and $-\int q\cot^2 p\,dp = -q\int(\operatorname{cosec}^2 p - 1)dp$

$= -q(-\cot p - p) = (p+\cot p)q$ —— ⑤

From ④ and ⑤ we write common val of F in ③, then

$F = (p+\cot p)q$ which is our required generating function.

④ $P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p$ and $Q = \ln(1+\sqrt{q}$

lution.

given that $P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p$

nd $Q = \ln(1+\sqrt{q}\cos p)$

efore, $dQ = \frac{1}{1+\sqrt{q}\cos p}\left(-\sqrt{q}\sin p\,dp + \frac{1}{2\sqrt{q}}\cos p\,dq\right)$

us $dF = p\,dq - 2\sqrt{q}\sin p\left(\frac{1}{2\sqrt{q}}\cos p\,dq - \sqrt{q}\sin p\,dp\right)$

$= p\,dq - \sin p\cos p\,dq + 2q\sin^2 p\,dp.$

$dF = \left(p - \frac{1}{2}\sin 2p\right)dq + 2q\sin^2 p\,dp. \quad \text{---(1)}$

$= M(q,p)\,dq + N(q,p)\,dp$

here $M = p - \frac{1}{2}\sin 2p$; $N = 2q\sin^2 p$.

$\Rightarrow \frac{\partial M}{\partial p} = 1 - \cos 2p$; $\frac{\partial N}{\partial q} = 2\sin^2 p$

$= 1 - \cos 2p$

$\Rightarrow \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$

ence the R.H.S of (1) is exact differential
f F, so the given transformation is canonical.
Now we find the generating function F by (1).

$dF = \left(p - \frac{1}{2}\sin 2p\right)dq + 2q\sin^2 p\,dp$

$= d\left(q\left(p - \frac{1}{2}\sin 2p\right)\right)$

ntegrating, we get,

$F = q\left(p - \frac{1}{2}\sin 2p\right)$

is our required generating function.

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(5) $Q = \frac{1}{p}$ and $P = qp^2$.

Solution.

Given that $Q = 1/p$; $P = qp^2$.

Therefore, $dQ = \frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp$.

$= 0 - \frac{1}{p^2}dp = -\frac{1}{p^2}dp$.

Thus $dF = pdq - qp^2\left(-\frac{1}{p^2}dp\right)$

$\Rightarrow dF = pdq + qdp$ ——— (1)

$= M(q,p)dq + N(q,p)dp$

where $M = p$ and $N = q$

$\Rightarrow \frac{\partial M}{\partial p} = 1$; $\frac{\partial N}{\partial q} = 1$.

$\Rightarrow \frac{\partial M}{\partial p} = \partial N/\partial q$

which shows that the R.H.S. of (1) is exa differential of F so given transformati is canonical.

Now we find generating function as

(1) $dF = d(pq)$

taking integral, we get,

$F = pq$

which is our required generating function

(6) $Q = p$ and $P = -q$

Solution.

$dF = pdq + qdq$ which is similar to above question.

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$= q\tan p$, $P = \ln \sin p$.

on.

That $Q = q\tan p$; $P = \ln \sin p$.

$dQ = \tan p\, dq + q\sec^2 p\, dp$

$dF = p\,dq - \ln \sin p\,(\tan p\, dq + q\sec^2 p\, dp)$.

$= p\,dq - \tan p \ln \sin p\, dq - q\sec^2 p \ln \sin p\, dp$.

$= (p - \tan p \ln \sin p)\, dq - q\sec^2 p \ln \sin p\, dp$ ①

$= M(q,p)\, dq + N(q,p)\, dp$

$M = p - \tan p \ln \sin p$; $N = -q\sec^2 p \ln \sin p$

$\frac{\partial M}{\partial p} = 1 - \sec^2 p \ln \sin p - \tan p \frac{1}{\sin p} \cos p$

$= 1 - \sec^2 p \ln \sin p - \tan p \cot p$

$= 1 - \sec^2 p \ln \sin p - 1$

$\frac{\partial M}{\partial p} = -\sec^2 p \ln \sin p$

$\frac{\partial N}{\partial q} = -\sec^2 p \ln \sin p$

$\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$ which shows that the R.H.S of ① is exact

ferential of function F, so given transformation

canonical.

ow we find generating function as

llows, $dF = d(q(p - \tan p \ln \sin p))$

aking integral, we get,

$F = q(p - \tan p \ln \sin p)$ which is our desired g-function.

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(8) $q = \sqrt{\frac{2P}{\sqrt{km}}} \sin Q$ and $p = \sqrt{2P\sqrt{km}} \cos Q$

Solution.

Given that

$$q = \sqrt{\frac{2P}{\sqrt{km}}} \sin Q \longrightarrow (1)$$

$$p = \sqrt{2P\sqrt{km}} \cos Q \longrightarrow (2)$$

Dividing (1) by (2), we get,

$$\frac{q}{p} = \sqrt{\frac{2P}{\sqrt{km}} \times \frac{1}{2P\sqrt{km}}} \frac{\sin Q}{\cos Q}$$

$$\Rightarrow \frac{q}{p} = \sqrt{\frac{1}{\sqrt{km}}} \tan Q = \frac{1}{\sqrt{km}} \tan Q.$$

Taking differential, we get,

$$\frac{dq}{p} - \frac{q}{p^2} dp = \frac{1}{\sqrt{km}} \sec^2 Q \, dQ.$$

$$\Rightarrow dQ = \sqrt{km} \left(\frac{dq}{p} - \frac{q}{p^2} dp \right) \cos^2 Q \quad \text{---} (3)$$

Squaring on both sides of (2), we get,

$$p^2 = 2P\sqrt{km} \cos^2 Q.$$

$$\Rightarrow P = \frac{1}{2\sqrt{km}} p^2 \sec^2 Q \quad \text{---} (4)$$

Thus $dF = p\,dq - P\,dQ$

$$= p\,dq - \frac{p^2}{2\sqrt{km}} \sec^2 Q \sqrt{km} \left(\frac{dq}{p} - \frac{q}{p^2} dp \right)$$

$$= p\,dq - \frac{p^2}{2} \left(\frac{dq}{p} - \frac{q}{p^2} dp \right)$$

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$= pdq - \frac{1}{2}pdq + \frac{1}{2}qdp$

$dF = \frac{1}{2}pdq + \frac{1}{2}qdp$ ——— (5)

$= M(q,p)dq + N(q,p)dp$

where $M = \frac{1}{2}p$; $N = \frac{1}{2}q$

$\Rightarrow \frac{\partial M}{\partial p} = 1/2$; $\frac{\partial N}{\partial q} = \frac{1}{2} \Rightarrow \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$.

Thus the R.H.S. of (5) is exact differential of function F so given transformation is Canonical.

Now we find generating function F as follows.

(5) $\Rightarrow dF = \frac{1}{2}d(pq)$

Taking integral, we get,

$F = \frac{1}{2}pq$ ——— (6)

To prove $F = \frac{1}{2}\sqrt{km}\, q^2 \cot Q$ we proceed as follows.

$\div$ (1) $\Rightarrow p/q = \sqrt{km}\cot Q$.

$\Rightarrow p = q\sqrt{km}\cot Q$

Now equation (6) becomes,

$F = \frac{1}{2}q(q\sqrt{km}\cot Q)$.

$\Rightarrow F = \frac{1}{2}\sqrt{km}\, q^2 \cot Q$ which is our required generating function. i.e. F_1.

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Poisson Bracket: 2009,

The poisson bracket of any two functions $F(t, q_s, p_s)$ and $G(t, q_s, p_s)$ with respect to the canonical variables (q, p) is written as $[F, G]_{q,p}$ and is defined by

$$[F,G]_{q,p} = \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial G}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial G}{\partial q_s} \right)$$

$$= \sum_{s=1}^{n} \begin{vmatrix} \frac{\partial F}{\partial q_s} & \frac{\partial F}{\partial p_s} \\ \frac{\partial G}{\partial q_s} & \frac{\partial G}{\partial p_s} \end{vmatrix} = \text{Sum of n Jacobi}$$

Fundmental Poisson Brackets:

① The poisson bracket of two (position) coordinates is z

i.e. $[q_j, q_k]_{q,p} = 0$

Proof.

$$[q_j, q_k]_{q,p} = \sum_{s=1}^{n} \left(\frac{\partial q_j}{\partial q_s} \frac{\partial q_k}{\partial p_s}^{0} - \frac{\partial q_j}{\partial p_s}^{0} \frac{\partial q_k}{\partial q_s} \right) = 0$$

② The poisson bracket of two momentum coord is zero.

i.e. $[p_j, p_k]_{q,p} = \sum_{s=1}^{n} \left(\frac{\partial p_j}{\partial q_s}^{0} \frac{\partial p_k}{\partial p_s} - \frac{\partial p_j}{\partial p_s} \frac{\partial p_k}{\partial q_s}^{0} \right) = 0$

③ The poisson bracket of position coordinate and momentum coordinate is Kronecker delt

i.e. $[q_j, p_k]_{q,p} = \delta_{jk}$

Proof

$$[q_j, p_k]_{q,p} = \sum_{s=1}^{n} \left(\frac{\partial q_j}{\partial q_s} \frac{\partial p_k}{\partial p_s} - \frac{\partial q_j}{\partial p_s} \frac{\partial p_k}{\partial q_s}^{0} \right)$$

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$$= \sum_{s=1}^{n} \left(\frac{\partial q_j}{\partial q_s} \frac{\partial p_k}{\partial p_s} \right) \quad \text{---------- (1)}$$

Since $\frac{\partial q_j}{\partial q_s} = \delta_{sj} = \begin{cases} 1 & \text{if } s = j \\ 0 & \text{if } s \neq j \end{cases}$

$\frac{\partial p_k}{\partial p_s} = \delta_{sk} = \begin{cases} 1 & \text{if } s = k \\ 0 & \text{if } s \neq k \end{cases}$

Therefore (1) implies

$$[q_j, p_k]_{q,p} = \sum_{s=1}^{n} \delta_{sj}\, \delta_{sk} = \delta_{jk} \qquad \because \delta_{sj}\delta_{sk} = \delta_{jk}.$$

$$\Rightarrow [q_j, p_k] = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Fundamental Properties of Poisson Brackets.

If the functions F, G, S depend on the position coordinates q_s, momenta p_s and time t. Then the fundamental identities can be obtained from the definition of Poisson bracket.

(1) $[F, F] = 0$

Proof.

Let $F = F(t, q_s, p_s)$, then by definition of Poisson bracket we can write,

$$[F, F] = \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial F}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial F}{\partial q_s} \right) = 0$$

(2) $[F, C] = 0$, where C is a constant.

Proof.

Let F be function of q_s, p_s and t, then

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$$[F,C] = \sum_{s=1}^{n}\left(\frac{\partial F}{\partial q_s}\overbrace{\frac{\partial C}{\partial p_s}}^{0} - \frac{\partial F}{\partial p_s}\overbrace{\frac{\partial C}{\partial q_s}}^{0}\right) = 0 \quad \because C \text{ is}$$

③ $[F,q] = -[q,F]$

Proof.

Let F, q be the functions of q_s, p_s and time

Then by Poisson bracket,

$$[F,q] = \sum_{s=1}^{n}\left(\frac{\partial F}{\partial q_s}\frac{\partial q}{\partial p_s} - \frac{\partial F}{\partial p_s}\frac{\partial q}{\partial q_s}\right)$$

$$= -\sum_{s=1}^{n}\left(\frac{\partial q}{\partial q_s}\frac{\partial F}{\partial p_s} - \frac{\partial q}{\partial p_s}\frac{\partial F}{\partial q_s}\right) = -[q,F$$

④ $[F, q+S] = [F,q] + [F,S]$.

Proof.

Let F, q, S be the functions of q_s, p_s an

then by definition of Poisson brackets,

$$[F, q+S] = \sum_{s=1}^{n}\left(\frac{\partial F}{\partial q_s}\frac{\partial (q+S)}{\partial p_s} - \frac{\partial F}{\partial p_s}\frac{\partial (q+S)}{\partial q_s}\right)$$

$$= \sum_{s=1}^{n}\left\{\frac{\partial F}{\partial q_s}\left(\frac{\partial q}{\partial p_s} + \frac{\partial S}{\partial p_s}\right) - \frac{\partial F}{\partial p_s}\left(\frac{\partial q}{\partial q_s} + \frac{\partial}{\partial}\right.\right.$$

$$= \sum_{s=1}^{n}\left(\frac{\partial F}{\partial q_s}\frac{\partial q}{\partial p_s} + \frac{\partial F}{\partial q_s}\frac{\partial S}{\partial p_s} - \frac{\partial F}{\partial p_s}\frac{\partial q}{\partial q_s} - \frac{\partial F}{\partial p_s}\right.$$

$$= \sum_{s=1}^{n}\left(\frac{\partial F}{\partial q_s}\frac{\partial q}{\partial p_s} - \frac{\partial F}{\partial p_s}\frac{\partial q}{\partial q_s}\right)$$

$$+ \sum_{s=1}^{n}\left(\frac{\partial F}{\partial q_s}\frac{\partial S}{\partial p_s} - \frac{\partial F}{\partial p_s}\frac{\partial}{\partial}\right.$$

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$\Rightarrow [F, G+S] = [F, G] + [F, S]$.

(5) $[F, GS] = [F, G]S + G[F, S]$.

Proof.

Let F, G, S be the functions of q_s, p_s and t then by definition of Poission bracket,

$$[F, GS] = \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial (GS)}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial (GS)}{\partial q_s} \right).$$

$$= \sum_{s=1}^{n} \left\{ \frac{\partial F}{\partial q_s} \left(\frac{\partial G}{\partial p_s} S + G \frac{\partial S}{\partial p_s} \right) - \frac{\partial F}{\partial p_s} \left(\frac{\partial G}{\partial q_s} S + G \frac{\partial S}{\partial q_s} \right) \right.$$

$$= \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial G}{\partial p_s} S + G \frac{\partial F}{\partial q_s} \frac{\partial S}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial G}{\partial q_s} S - G \frac{\partial F}{\partial p_s} \frac{\partial S}{\partial q_s} \right.$$

$$= S \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial G}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial G}{\partial q_s} \right) + G \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial S}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial S}{\partial q_s} \right.$$

$$= S[F, G] + G[F, S]$$

(6) $\frac{d}{dt}[F, G] = \left[\frac{dF}{dt}, G\right] + \left[F, \frac{dG}{dt}\right]$

Proof.

Let F, G be functions of q_s, p_s and t then by definition of Poisson bracket,

$$[F, G] = \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial G}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial G}{\partial q_s} \right)$$

$$\Rightarrow \frac{d}{dt}[F, G] = \frac{d}{dt} \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial G}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial G}{\partial q_s} \right)$$

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$$= \sum_{s=1}^{n} \frac{d}{dt}\left(\frac{\partial F}{\partial q_s}\frac{\partial G}{\partial p_s}\right) - \sum_{s=1}^{n} \frac{d}{dt}\left(\frac{\partial F}{\partial p_s}\frac{\partial G}{\partial q_s}\right)$$

$$= \sum_{s=1}^{n}\left\{\frac{d}{dt}\left(\frac{\partial F}{\partial q_s}\right)\frac{\partial G}{\partial p_s} + \frac{\partial F}{\partial q_s}\frac{d}{dt}\left(\frac{\partial G}{\partial p_s}\right)\right\}$$

$$- \sum_{s=1}^{n}\left\{\frac{d}{dt}\left(\frac{\partial F}{\partial p_s}\right)\frac{\partial G}{\partial q_s} + \frac{\partial F}{\partial p_s}\frac{d}{dt}\left(\frac{\partial G}{\partial q_s}\right)\right\}$$

$$= \sum_{s=1}^{n}\left\{\frac{\partial}{\partial q_s}\left(\frac{dF}{dt}\right)\frac{\partial G}{\partial p_s} + \frac{\partial F}{\partial q_s}\frac{\partial}{\partial p_s}\left(\frac{dG}{dt}\right)\right\}$$

$$- \sum_{s=1}^{n}\left\{\frac{\partial}{\partial p_s}\left(\frac{dF}{dt}\right)\frac{\partial G}{\partial q_s} + \frac{\partial F}{\partial p_s}\frac{\partial}{\partial q_s}\left(\frac{dG}{dt}\right)\right\}$$

$$= \sum_{s=1}^{n}\left\{\frac{\partial}{\partial q_s}\left(\frac{dF}{dt}\right)\frac{\partial G}{\partial p_s} - \frac{\partial}{\partial p_s}\left(\frac{dF}{dt}\right)\frac{\partial G}{\partial q_s}\right\}$$

$$+ \sum_{s=1}^{n}\left\{\frac{\partial F}{\partial q_s}\frac{\partial}{\partial p_s}\left(\frac{dG}{dt}\right) - \frac{\partial F}{\partial p_s}\frac{\partial}{\partial q_s}\left(\frac{dG}{dt}\right)\right\}$$

$$= \left[\frac{dF}{dt}, G\right] + \left[F, \frac{dG}{dt}\right].$$

2009

⑦ $$\frac{\partial}{\partial t}[F, G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right]$$

Proof.

Similar as above.

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9/1/2015

Jacobi identity for Poisson Bracket.

$$[F,[G,S]]+[G,[S,F]]+[S,[F,G]]=0$$

Proof:

Let F, G, S be functions of q_s, p_s and t, then by Poisson bracket,

$$[G,S]=\sum_{s=1}^{n}\left(\frac{\partial G}{\partial q_s}\frac{\partial S}{\partial p_s}-\frac{\partial G}{\partial p_s}\frac{\partial S}{\partial q_s}\right)$$

$$[F,[G,S]]=\sum_{r=1}^{n}\left\{\frac{\partial F}{\partial q_r}\left(\frac{\partial}{\partial p_r}\sum_{s=1}^{n}\left(\frac{\partial G}{\partial q_s}\frac{\partial S}{\partial p_s}-\frac{\partial G}{\partial p_s}\frac{\partial S}{\partial q_s}\right)\right)-\frac{\partial F}{\partial p_r}\left(\frac{\partial}{\partial q_r}\sum_{s=1}^{n}\left(\frac{\partial G}{\partial q_s}\frac{\partial S}{\partial p_s}-\frac{\partial G}{\partial p_s}\frac{\partial S}{\partial q_s}\right)\right)\right\}$$

$$=\sum_{r=1}^{n}\sum_{s=1}^{n}\left[\frac{\partial F}{\partial q_r}\left\{\frac{\partial}{\partial p_r}\left(\frac{\partial G}{\partial q_s}\frac{\partial S}{\partial p_s}\right)\right\}-\frac{\partial F}{\partial q_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial G}{\partial p_s}\frac{\partial S}{\partial q_s}\right)\right\}-\frac{\partial F}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial G}{\partial q_s}\frac{\partial S}{\partial p_s}\right)\right\}+\frac{\partial F}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial G}{\partial p_s}\frac{\partial S}{\partial q_s}\right)\right\}\right]\longrightarrow(1)$$

Similarly, by symmetry, we can write,

$$[G,[S,F]]=\sum_{r=1}^{n}\sum_{s=1}^{n}\left[\frac{\partial G}{\partial q_r}\left\{\frac{\partial}{\partial p_r}\left(\frac{\partial S}{\partial q_s}\frac{\partial F}{\partial p_s}\right)\right\}-\frac{\partial G}{\partial q_r}\left\{\frac{\partial}{\partial p_r}\left(\frac{\partial S}{\partial p_s}\frac{\partial F}{\partial q_s}\right)\right\}-\frac{\partial G}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial S}{\partial q_s}\frac{\partial F}{\partial p_s}\right)\right\}+\frac{\partial G}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial S}{\partial p_s}\frac{\partial F}{\partial q_s}\right)\right\}\right]\longrightarrow(2)$$

and

$$[S,[F,G]]=\sum_{r=1}^{n}\sum_{s=1}^{n}\left[\frac{\partial S}{\partial q_r}\left\{\frac{\partial}{\partial p_r}\left(\frac{\partial F}{\partial q_s}\frac{\partial G}{\partial p_s}\right)\right\}-\frac{\partial S}{\partial q_r}\left\{\frac{\partial}{\partial p_r}\left(\frac{\partial F}{\partial p_s}\frac{\partial G}{\partial q_s}\right)\right\}-\frac{\partial S}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial F}{\partial q_s}\frac{\partial G}{\partial p_s}\right)\right\}+\frac{\partial S}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial F}{\partial p_s}\frac{\partial G}{\partial q_s}\right)\right\}\right]\longrightarrow(3)$$

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Don't worry

HaHaHa

Adding ①, ② and ③, we get,

$$[F,[G,S]]+[G,[S,F]]+[S,[F,G]]=\sum_{r=1}^{n}\sum_{s=1}^{n}\left[\frac{\partial F}{\partial q_r}\left\{\frac{\partial}{\partial p_r}\left(\frac{\partial}{\partial}\right.\right.\right.$$

$$-\frac{\partial F}{\partial q_r}\left(\frac{\partial}{\partial p_r}\left(\frac{\partial G}{\partial q_s}\frac{\partial S}{\partial p_s}\right)\right)-\frac{\partial F}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial G}{\partial q_s}\frac{\partial S}{\partial p_s}\right)\right\}+\frac{\partial F}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial G}{\partial q_s}\right.\right.$$

$$+\frac{\partial G}{\partial q_r}\left\{\frac{\partial}{\partial p_r}\left(\frac{\partial S}{\partial q_s}\frac{\partial F}{\partial p_s}\right)\right\}-\frac{\partial G}{\partial q_r}\left\{\frac{\partial}{\partial p_r}\left(\frac{\partial S}{\partial q_s}\frac{\partial F}{\partial p_s}\right)\right\}-\frac{\partial G}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial S}{\partial q_s}\right.\right.$$

$$+\frac{\partial G}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial S}{\partial q_s}\frac{\partial F}{\partial p_s}\right)\right\}+\frac{\partial S}{\partial q_r}\left\{\frac{\partial}{\partial p_r}\left(\frac{\partial F}{\partial q_s}\frac{\partial G}{\partial p_s}\right)\right\}-\frac{\partial S}{\partial q_r}\left\{\frac{\partial}{\partial p_r}\left(\frac{\partial F}{\partial q_s}\right.\right.$$

$$-\frac{\partial S}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial F}{\partial q_s}\frac{\partial G}{\partial p_s}\right)\right\}+\frac{\partial S}{\partial p_r}\left\{\frac{\partial}{\partial q_r}\left(\frac{\partial F}{\partial q_s}\frac{\partial G}{\partial p_s}\right)\right\}=0$$

$$\Rightarrow [F,[G,S]]+[G,[S,F]]+[S,[F,G]]=0$$

Equations of Motion in Poisson Bracket Form.

2018

(OR) If H is Hamiltonian, prove that if any function depending on position, momenta and time then $\frac{dF}{dt}=\frac{\partial F}{\partial t}+[H,F]$.

Let F be the function of q_s, p_s and t then

$$\frac{dF}{dt}=\frac{\partial F}{\partial t}\frac{dt}{dt}+\sum_{s=1}^{n}\left(\frac{\partial F}{\partial q_s}\frac{dq_s}{dt}+\frac{\partial F}{\partial p_s}\frac{dp_s}{dt}\right)$$

$$=\frac{\partial F}{\partial t}+\sum_{s=1}^{n}\left(\frac{\partial F}{\partial q_s}\dot{q}_s+\frac{\partial F}{\partial p_s}\dot{p}_s\right)\quad\text{---}(4)$$

By Hamilton's equations of motion,

$$\dot{p}_s=-\frac{\partial H}{\partial q_s}\quad\text{and}\quad\dot{q}_s=\frac{\partial H}{\partial p_s}$$

Now equation ④ becomes

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$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s}\frac{\partial H}{\partial p_s} - \frac{\partial F}{\partial p_s}\frac{\partial H}{\partial q_s} \right)$$

By definition of Poisson bracket,

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H] \quad \text{---} \quad (2)$$

which is known as equation of motion of F in terms of Poisson Bracket.

If we replace F by q_s and p_s in equation (2) when q_s and p_s do not depend on t explicitly.

Then

$$\frac{dq_s}{dt} = \frac{\partial q_s}{\partial t}^{0} + [q_s, H] \qquad \because q_s \neq q_s(t).$$

$$\Rightarrow \dot{q}_s = [q_s, H] \quad \text{where } H \text{ is Hamiltonian.}$$

and $$\frac{dp_s}{dt} = \frac{\partial p_s}{\partial t}^{0} + [p_s, H] \qquad \because p_s \neq p_s(t).$$

$$\Rightarrow \dot{p}_s = [p_s, H].$$

These two equations constitute the canonical equations of motion in Poisson bracket form.

Poisson's Theorem. 2009, 2012, 2013, 2016,

If $F(q, p, t)$ and $G(q, p, t)$ are two integrals of motion, then $[F, G]$ is also an integral of motion.

i.e. $[F, G] = \text{Constant}$.

(OR)

"If u and v are any two constant of motion of any given holonomic dynamical system, then show Poisson bracket $[u, v]$ is also a constant of motion."

Proof.

To show $[F, G] = \text{Constant}$, we show that use $[u, v]$ in terms of $[F, G]$

$$\frac{d}{dt}[F,G] = 0$$

Since $F(t,q,p)$ and $G(t,q,p)$ are integrals of motion.

Therefore, $\frac{dF}{dt} = 0$ and $\frac{dG}{dt} = 0$

By equation of motion in Poisson bracket

$$\frac{\partial F}{\partial t} + [F,H] = 0$$

$$\Rightarrow \frac{\partial F}{\partial t} = -[F,H] = [H,F] \quad \text{——} (1)$$

and $$\frac{\partial G}{\partial t} + [G,H] = 0$$

$$\Rightarrow \frac{\partial G}{\partial t} = -[G,H] = [H,G] \quad \text{——} (2)$$

Taking time derivate of the equation of m in the Poisson bracket form, we get,

$$\frac{d}{dt}[F,G] = \frac{\partial [F,G]}{\partial t} + [[F,G],H]$$

$$= \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right] + [[F,G],H]$$

Subsituting (1) and (2), in above equation we get,

$$\frac{d}{dt}[F,G] = [[H,F],G] + [F,[H,G]] + [[F,G],H]$$

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$= [[H,F],G] - [[H,G],F] + [[F,G],H]$ using J Prop.

$= [[H,F],G] + [[G,H],F] + [[F,G],H]$

$= [[H,F],G] + [[F,G],H] + [[G,H],F]$

$= 0$ ∵ by Jacobi identity.

$\frac{d}{dt}[F,G] = 0$

$\Rightarrow [F,G] =$ Constant.

Hence proved.

Question. 2017, 2018, :—

If F and G are any pair of functions of (q,p) or (Q,P), then show that the Poisson bracket of the functions F and G is invariant under canonical transformation. (OR) Show poisson bracket is invariance under canonical transformation

Solution.

Let F and G be any pair of functions of (q,p) or (Q,P), the sets (q,p) and (Q,P) are related by a canonical transformation which is given by

$Q_i = Q_i(t,q,p)$ and $P_i = P_i(t,q,p)$

then the Poisson bracket of the functions F and G with respect to the (q,p) set is given by

$$[F,G]_{q,p} = \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s}\frac{\partial G}{\partial p_s} - \frac{\partial F}{\partial p_s}\frac{\partial G}{\partial q_s} \right) \longrightarrow (1)$$

Consider the function $G = G(Q_i, P_i)$

Then, by chain rule

$$\frac{\partial G}{\partial p_s} = \sum_{i=1}^{n} \left(\frac{\partial G}{\partial Q_i} \frac{\partial Q_i}{\partial p_s} + \frac{\partial G}{\partial P_i} \frac{\partial P_i}{\partial p_s} \right) \longrightarrow (2)$$

$$\text{and } \frac{\partial G}{\partial q_s} = \sum_{i=1}^{n} \left(\frac{\partial G}{\partial Q_i} \frac{\partial Q_i}{\partial q_s} + \frac{\partial G}{\partial P_i} \frac{\partial P_i}{\partial q_s} \right) \longrightarrow (3)$$

Subsituting (2) and (3) in (1), we get,

$$[F,G]_{q,p} = \sum_{s=1}^{n} \left[\frac{\partial F}{\partial q_s} \sum_{i=1}^{n} \left(\frac{\partial G}{\partial Q_i} \frac{\partial Q_i}{\partial p_s} + \frac{\partial G}{\partial P_i} \frac{\partial P_i}{\partial p_s} \right) - \frac{\partial F}{\partial p_s} \sum_{i=1}^{n} \left(\frac{\partial G}{\partial Q_i} \frac{\partial Q_i}{\partial q_s} + \frac{\partial G}{\partial P_i} \frac{\partial P_i}{\partial q_s} \right) \right]$$

$$= \sum_{s=1}^{n} \sum_{i=1}^{n} \left[\frac{\partial G}{\partial Q_i} \left(\frac{\partial F}{\partial q_s} \frac{\partial Q_i}{\partial p_s} \right) + \frac{\partial G}{\partial P_i} \left(\frac{\partial F}{\partial q_s} \frac{\partial P_i}{\partial p_s} \right) - \frac{\partial G}{\partial Q_i} \left(\frac{\partial F}{\partial p_s} \frac{\partial Q_i}{\partial q_s} \right) - \frac{\partial G}{\partial P_i} \left(\frac{\partial F}{\partial p_s} \frac{\partial P_i}{\partial q_s} \right) \right]$$

$$= \sum_{s=1}^{n} \sum_{i=1}^{n} \left[\frac{\partial G}{\partial Q_i} \left(\frac{\partial F}{\partial q_s} \frac{\partial Q_i}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial Q_i}{\partial q_s} \right) + \frac{\partial G}{\partial P_i} \left(\frac{\partial F}{\partial q_s} \frac{\partial P_i}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial P_i}{\partial q_s} \right) \right]$$

$$= \sum_{i=1}^{n} \left[\frac{\partial G}{\partial Q_i} \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial Q_i}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial Q_i}{\partial q_s} \right) + \frac{\partial G}{\partial P_i} \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial P_i}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial P_i}{\partial q_s} \right) \right]$$

$$= \sum_{i=1}^{n} \left[\frac{\partial G}{\partial Q_i} [F, Q_i]_{q,p} + \frac{\partial G}{\partial P_i} [F, P_i]_{q,p} \right] \longrightarrow (4)$$

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placing F by Q_K and G by F, we have

$$[F] = \sum_{i=1}^{n} \left[\frac{\partial F}{\partial Q_i} [Q_K, Q_i]_{q,p} + \frac{\partial F}{\partial P_i} [Q_K, P_i]_{q,p} \right] \longrightarrow (5)$$

since $[Q_K, Q_i] = 0$ and $[Q_K, P_i] = \delta_{iK}$

$$[Q_K, F] = \frac{\partial F}{\partial P_K} \Rightarrow [F, Q_K] = -\frac{\partial F}{\partial P_K}$$

In the same way from equation (4), we can write,

$$[F, Q_i] = -\frac{\partial F}{\partial P_i} \quad \text{and} \quad [F, P_i] = \frac{\partial F}{\partial Q_i}$$

Now equation (4) becomes

$$[F, G]_{q,p} = \sum_{i=1}^{n} \left[\frac{\partial G}{\partial Q_i} \left(-\frac{\partial F}{\partial P_i} \right) + \frac{\partial G}{\partial P_i} \left(\frac{\partial F}{\partial Q_i} \right) \right]$$

$$= \sum_{i=1}^{n} \left(\frac{\partial F}{\partial Q_i} \frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i} \frac{\partial G}{\partial Q_i} \right)$$

$$\Rightarrow [F, G]_{q,p} = [F, G]_{Q,P}$$

Thus Poisson brackets are invariant under canonical transformation.

Exercise.

Show that (i) $[Q_K, Q_i] = 0$ (ii) $[Q_K, P_i] = \delta_{iK}$

(iii) $[F, Q_i] = -\frac{\partial F}{\partial P_i}$ (iv) $[F, P_i] = \frac{\partial F}{\partial Q_i}$

Solution.

Here Q_i, P_i are two canonical variables.

i.e. each $Q_i = Q_i(t, q_s, p_s)$ and $P_i = P_i(t, q_s, p_s)$

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$\Rightarrow q_s = q_s(t, Q_i, P_i)$ and $p_s = p_s(t, Q_i, P_i)$.

Let $F = F(t, q_s, p_s)$ and $G = G(t, q_s, p_s)$

$\Rightarrow F = F(t, Q_s, P_s)$ and $G = G(t, Q_s, P_s)$.

Thus $[F, G]_{q,p} = \sum_{s=1}^{n} \left(\frac{\partial F}{\partial q_s} \frac{\partial G}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial G}{\partial q_s} \right)$

$\Rightarrow [F, G]_{Q,P} = \sum_{s=1}^{n} \left(\frac{\partial F}{\partial Q_s} \frac{\partial G}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial G}{\partial Q_s} \right)$ — (1)

(i) $[Q_k, Q_i] = \sum_{s=1}^{n} \left(\frac{\partial Q_k}{\partial Q_s} \frac{\partial Q_i}{\partial p_s}^{0} - \frac{\partial Q_k}{\partial p_s}^{0} \frac{\partial Q_i}{\partial Q_s} \right) = 0$

(ii) $[Q_k, P_i] = \sum_{s=1}^{n} \left(\frac{\partial Q_k}{\partial Q_s} \frac{\partial P_i}{\partial p_s} - \frac{\partial Q_k}{\partial p_s} \frac{\partial P_i}{\partial Q_s}^{0} \right)$

$= \sum_{s=1}^{n} \left(\frac{\partial Q_k}{\partial Q_s} \frac{\partial P_i}{\partial p_s} \right) = \sum_{s=1}^{n} \delta_{sk} \delta_{si} = \delta_{ik}$

(iii) $[F, Q_i] = \sum_{s=1}^{n} \left(\frac{\partial F}{\partial Q_s} \frac{\partial Q_i}{\partial p_s}^{0} - \frac{\partial F}{\partial p_s} \frac{\partial Q_i}{\partial Q_s} \right)$

$= -\frac{\partial F}{\partial p_i}$ when $i = s$

(iv) $[F, P_i] = \sum_{s=1}^{n} \left(\frac{\partial F}{\partial Q_s} \frac{\partial P_i}{\partial p_s} - \frac{\partial F}{\partial p_s} \frac{\partial P_i}{\partial Q_s}^{0} \right)$

$= \frac{\partial F}{\partial Q_i}$ when $i = s$.

Example 1 :—

Using the Poisson bracket, show the tr

$q = \sqrt{2p} \sin Q$, $P = \sqrt{2p} \cos Q$

is Canonical.

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Solution.

By definition of Poisson bracket, it is obvious that

$[Q, Q] = 0$ and $[P, P] = 0$.

From the given data, we have,

$$q = \sqrt{2P}\sin Q \quad \text{——— (1)}$$

$$p = \sqrt{2P}\cos Q \quad \text{——— (2)}$$

we show that $[Q, P] = 1$

$$[Q, P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}; s=1 \quad \text{——— (3)}$$

Squaring (1) and (2) and then adding, we get,

$$q^2 + p^2 = 2P\sin^2 Q + 2P\cos^2 Q.$$

$$\Rightarrow q^2 + p^2 = 2P$$

$$\Rightarrow P = \frac{1}{2}(q^2 + p^2)$$

$$\Rightarrow \frac{\partial P}{\partial p} = p \quad \text{and} \quad \frac{\partial P}{\partial q} = q \quad \text{——— (4)}$$

Dividing (1) by (2), we get

$$\tan Q = q/p$$

$$\Rightarrow Q = \tan^{-1}(q/p)$$

$$\Rightarrow \frac{\partial Q}{\partial p} = \frac{-q}{p^2+q^2} \quad \text{and} \quad \frac{\partial Q}{\partial q} = \frac{p}{p^2+q^2} \quad \text{——— (5)}$$

Subsituding equation (4) and (5) in (3), we get,

$$[Q, P] = \frac{p^2}{p^2+q^2} + \frac{q^2}{p^2+q^2} = \frac{p^2+q^2}{p^2+q^2} = 1$$

Hence, the transformation is canonical.

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Example 2:-
Use the Poisson bracket to show that the transformation

$$Q = \ln(1+\sqrt{q}\cos p)$$
$$P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p$$

is canonical or contact.

Solution.

Given that

$$P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p \quad \text{---} (1)$$
$$Q = \ln(1+\sqrt{q}\cos p) \quad \text{---} (2)$$

By definition of Poisson bracket. it is obvious that $[Q,Q]=0$ and $[P,P]=0$

we show that $[Q,P]=1$.

Now, $[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$ ---

$$(1) \Rightarrow P = 2\sqrt{q}\sin p + 2q\sin p\cos p$$
$$= 2\sqrt{q}\sin p + q\sin 2p$$
$$\Rightarrow \frac{\partial P}{\partial p} = 2\sqrt{q}\cos p + 2q\cos 2p \quad \text{---}$$

and $$\frac{\partial P}{\partial q} = \frac{1}{\sqrt{q}}\sin p + \sin 2p \quad \text{---} (5)$$

$$(2) \Rightarrow \frac{\partial Q}{\partial p} = \frac{-\sqrt{q}\sin p}{1+\sqrt{q}\cos p} \quad \text{---} (6)$$

and $$\frac{\partial Q}{\partial q} = \frac{\cos p}{2\sqrt{q}(1+\sqrt{q}\cos p)} \quad \text{---} (7)$$

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Thus equation (3) becomes

$$[Q,P] = \frac{\cos p(2\sqrt{q}\cos p + 2q\cos 2p)}{2\sqrt{q}(1+\cos p\sqrt{q})} + \frac{\sqrt{q}\sin p\left(\frac{1}{\sqrt{q}}\sin p + \sin 2p\right)}{1+\sqrt{q}\cos p}$$

$$= \frac{\cos^2 p + \sqrt{q}\cos p\cos 2p}{1+\sqrt{q}\cos p} + \frac{\sin^2 p + \sqrt{q}\sin p\sin 2p}{1+\sqrt{q}\cos p}$$

$$= \frac{\cos^2 p + \sqrt{q}\cos p\cos 2p + \sin^2 p + \sqrt{q}\sin p\sin 2p}{1+\sqrt{q}\cos p}$$

$$= \frac{1 + \sqrt{q}\cos p(\cos^2 p - \sin^2 p) + 2\sqrt{q}\cos p\sin^2 p}{1+\sqrt{q}\cos p}$$

$$= \frac{1+\sqrt{q}\cos p(\cos^2 p - \sin^2 p + 2\sin^2 p)}{1+\sqrt{q}\cos p}$$

$$= \frac{1+\sqrt{q}\cos p(\sin^2 p + \cos^2 p)}{1+\sqrt{q}\cos p}$$

$$= \frac{1+\sqrt{q}\cos p}{1+\sqrt{q}\cos p} = 1$$

Hence our given transformation is canonical

Example 3. :-

Using Poisson bracket, show that the following transformation

$Q = \sqrt{e^{-2q} - p^2}$, $P = \cos^{-1}(pe^q)$

is canonical.

Solution.

Given that $Q = \sqrt{e^{-2q} - p^2}$ ——— (1)

$P = \cos^{-1}(pe^q)$ ——— (2)

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From definition of Poisson bracket, it is obvious that

$[Q,Q] = 0$ and $[P,P] = 0$

We show that $[Q,P] = 1$.

Now, $[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$ ——— (3)

① $\Rightarrow \frac{\partial Q}{\partial p} = \frac{1}{2\sqrt{e^{-2q}-p^2}}(-2p) = \frac{-p}{\sqrt{e^{-2q}-p^2}}$ ———

and $\frac{\partial Q}{\partial q} = \frac{1}{2\sqrt{e^{-2q}-p^2}}(-2e^{-2q}) = \frac{-e^{-2q}}{\sqrt{e^{-2q}-p^2}}$ ———

② $\Rightarrow \frac{\partial P}{\partial p} = \frac{-1}{\sqrt{1-(pe^q)^2}}(e^q)$ $\quad \because \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$

$= \frac{-e^q}{\sqrt{1-p^2e^{2q}}}$ ——— (6)

and $\frac{\partial P}{\partial q} = \frac{-1}{\sqrt{1-(pe^q)^2}}(pe^q) = \frac{-pe^q}{\sqrt{1-p^2e^{2q}}}$

Thus equation (3) becomes

$[Q,P] = \left(\frac{-e^{-2q}}{\sqrt{e^{-2q}-p^2}}\right)\left(\frac{-e^q}{\sqrt{1-p^2e^{2q}}}\right) - \left(\frac{-p}{\sqrt{e^{-2q}-p^2}}\right)\left(\frac{-pe^q}{\sqrt{1-p^2e^{2q}}}\right)$

$= \frac{e^{-q}}{\sqrt{e^{-2q}-p^2}\sqrt{e^{2q}(e^{-2q}-p^2)}} - \frac{p^2e^q}{\sqrt{e^{-2q}-p^2}\sqrt{e^{2q}(e^{-2q}-p^2)}}$

$= \frac{e^{-q}-p^2e^q}{e^q\sqrt{e^{-2q}-p^2}\sqrt{e^{-2q}-p^2}}$

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$$= \frac{e^{q}(e^{-2q} - p^2)}{e^{q}(e^{-2q} - p^2)} = 1.$$

Hence the given transformation is canonical.

Question:-

Given the canonical transformations $Q = \frac{1}{2}(q^2 + p^2)$, $P = -\tan^{-1}(q/p)$. Evaluate $[Q, P]$ and show $[H, [Q, P]] = 0$.

Solution.

Given that $P = -\tan^{-1}(q/p)$ ——— (1)

$$Q = \frac{1}{2}(p^2 + q^2) \quad \text{——— (2)}$$

By definition of Poisson bracket,

$$[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \quad \text{——— (3)}$$

$$(1) \Rightarrow \frac{\partial P}{\partial p} = \frac{q}{p^2+q^2}, \quad \frac{\partial P}{\partial q} = \frac{-p}{p^2+q^2}$$

$$(2) \Rightarrow \frac{\partial Q}{\partial p} = p \; ; \; \frac{\partial Q}{\partial q} = q$$

Now equation (3) becomes

$$[Q, P] = \frac{q^2}{p^2+q^2} + \frac{p^2}{p^2+q^2} = \frac{p^2+q^2}{p^2+q^2} = 1$$

Now $[H, [Q, P]] = [H, 1] = 0$

Since $[H, C] = 0$ where C is any constant.

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Question:-

Use the Poisson bracket to show that the transformation

$Q = \sqrt{p^2+q^2}$; $P = -\sqrt{p^2+q^2}\ \tan^{-1}(q/p)$

is canonical.

Solution.

Given that $Q = \sqrt{p^2+q^2}$ ——— (1)

$P = -\sqrt{p^2+q^2}\ \tan^{-1}(q/p)$ ——— (2)

By definition of Poisson bracket, it is obvious that $[Q,Q] = 0$, $[P,P] = 0$.

We show that $[Q,P] = 1$

Now $[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$ ——— (3)

(1) $\Rightarrow \frac{\partial Q}{\partial p} = \frac{p}{\sqrt{p^2+q^2}}$ and $\frac{\partial Q}{\partial q} = \frac{q}{\sqrt{p^2+q^2}}$

(2) $\Rightarrow \frac{\partial P}{\partial p} = \frac{-p}{\sqrt{p^2+q^2}}\tan^{-1}\left(\frac{q}{p}\right) + \frac{q}{\sqrt{p^2+q^2}}$

and $\frac{\partial P}{\partial q} = \frac{-q}{\sqrt{p^2+q^2}}\tan^{-1}\left(\frac{q}{p}\right) - \frac{p}{\sqrt{p^2+q^2}}$

Thus equation (3) becomes

$$[Q,P] = \frac{q}{\sqrt{p^2+q^2}}\left(\frac{-p}{\sqrt{p^2+q^2}}\tan^{-1}\left(\frac{q}{p}\right) + \frac{q}{\sqrt{p^2+q^2}}\right) - \frac{p}{\sqrt{p^2+q^2}}\left(\frac{-q}{\sqrt{p^2+q^2}}\tan^{-1}(q/p) - \frac{p}{\sqrt{p^2+q^2}}\right)$$

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$$\frac{1}{p^2+q^2}\left(-pq\tan^{-1}(q/p)+q^2+qp\tan^{-1}(q/p)+p^2\right)$$

$$\frac{p^2+q^2}{p^2+q^2}=1$$

ince given transformation is canonical.

Lagrange Brackets:-

he Lagrange bracket of any two functions $F(q,p)$ nd $G(q,p)$ w.r.t. (q,p) variables, written as $\{F,G\}_{q,p}$, is defined as

$$\{F,G\}_{q,p}=\sum_{s=1}^{n}\left(\frac{\partial q_s}{\partial F}\frac{\partial p_s}{\partial G}-\frac{\partial p_s}{\partial F}\frac{\partial q_s}{\partial G}\right)$$

perties.

$$\{F,G\}_{q,p}=-\{G,F\}_{q,p}$$

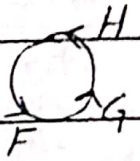
$$\{q_k,q_l\}=0$$

$$\{p_k,p_l\}=0$$

$$\{p_k,q_l\}=\delta_{kl}$$

The Jacobi's identity does not hold in Lagrange brackets.

$$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}\neq 0$$



نوٹ:۔ اگر یہ ثابت کرنا آجائے Poisson bracket کو zero کے برابر کرنا ہے۔ end

Awan
06-11-18

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Theorem:

The Lagrange brackets are invariant under Canonical transformation.

i.e. $\{F, G\}_{q,p} = \{F, G\}_{Q,P}$

Proof.

Let us suppose that a transformation from (q, p) to (Q, P) is a canonical transformation so that

$Q_i = Q_i(t, q_s, p_s)$ and $P_i = P_i(t, q_s, p_s)$

where $s = 1, 2, 3, \ldots n$

We assume that generating function in this case is of the form

$$F_1 = F_1(t, q_s, Q_s) \quad \text{---------- (1)}$$

which satisfies the conditions

$$p_s = \frac{\partial F_1}{\partial q_s} \quad \text{---------- (2)}$$

$$P_s = -\frac{\partial F_1}{\partial Q_s} \quad \text{---------- (3)}$$

Let us consider the Lagrange's bracket in this case when F, G are functions of q_s, p_s

$$\{F, G\}_{q,p} = \sum_{s=1}^{n} \left(\frac{\partial q_s}{\partial F} \frac{\partial p_s}{\partial G} - \frac{\partial p_s}{\partial F} \frac{\partial q_s}{\partial G} \right) \quad \text{----------}$$

$$\Rightarrow \{F, G\}_{q,p} = \sum_{s=1}^{n} \begin{vmatrix} \frac{\partial q_s}{\partial F} & \frac{\partial p_s}{\partial F} \\ \frac{\partial q_s}{\partial G} & \frac{\partial p_s}{\partial G} \end{vmatrix}$$

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$$\{F,G\}_{q,p} = \begin{vmatrix} \frac{\partial q_s}{\partial F} & \frac{\partial p_s}{\partial F} \\ \frac{\partial q_s}{\partial G} & \frac{\partial p_s}{\partial G} \end{vmatrix}$$ (by E.S.C) ——— (5)

Now $\frac{\partial p_s}{\partial F} = \frac{\partial}{\partial F}\left(\frac{\partial F_1}{\partial q_s}\right)$ ∵ by (2)

$\Rightarrow \frac{\partial p_s}{\partial F} = \frac{\partial}{\partial q_s}\left(\frac{\partial F_1}{\partial F}\right)$ ——— (6)

But $F_1 = F_1(t, q_s, Q_s)$

$\Rightarrow \frac{\partial F_1}{\partial F} = \frac{\partial F_1}{\partial t}\frac{\partial t}{\partial F}^{0} + \frac{\partial F_1}{\partial q_s}\frac{\partial q_s}{\partial F} + \frac{\partial F_1}{\partial Q_s}\frac{\partial Q_s}{\partial F}$ (by E.S.C)

$\Rightarrow \frac{\partial F_1}{\partial F} = \frac{\partial F_1}{\partial q_s}\frac{\partial q_s}{\partial F} + \frac{\partial F_1}{\partial Q_s}\frac{\partial Q_s}{\partial F}$

s is dummy index which can be changed by s

(6) $\Rightarrow \frac{\partial p_s}{\partial F} = \frac{\partial}{\partial q_s}\left(\frac{\partial F_1}{\partial q_s}\frac{\partial q_s}{\partial F}\right) + \frac{\partial}{\partial q_s}\left(\frac{\partial F_1}{\partial Q_s}\frac{\partial Q_s}{\partial F}\right)$

$\Rightarrow \frac{\partial p_s}{\partial F} = \frac{\partial^2 F_1}{\partial q_s \partial q_s}\frac{\partial q_s}{\partial F} + \frac{\partial^2 F_1}{\partial q_s \partial Q_s}\frac{\partial Q_s}{\partial F}$ ——— (7)

Similarly we can find

$\frac{\partial p_s}{\partial G} = \frac{\partial^2 F_1}{\partial q_s \partial q_s}\frac{\partial q_s}{\partial G} + \frac{\partial^2 F_1}{\partial q_s \partial Q_s}\frac{\partial Q_s}{\partial G}$ ——— (8)

Substituting relations (7) and (8) in (5), we have,

$$\{F,G\}_{q,p} = \begin{vmatrix} \frac{\partial q_s}{\partial F} & \frac{\partial^2 F_1}{\partial q_s \partial q_s}\frac{\partial q_s}{\partial F} + \frac{\partial^2 F_1}{\partial q_s \partial Q_s}\frac{\partial Q_s}{\partial F} \\ \frac{\partial q_s}{\partial G} & \frac{\partial^2 F_1}{\partial q_s \partial q_s}\frac{\partial q_s}{\partial G} + \frac{\partial^2 F_1}{\partial q_s \partial Q_s}\frac{\partial Q_s}{\partial G} \end{vmatrix}$$

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$$= \begin{vmatrix} \frac{\partial q_s}{\partial F} & \frac{\partial^2 F_1}{\partial q_s \partial q_\ell} & \frac{\partial q_\ell}{\partial F} \\ \frac{\partial q_s}{\partial G} & \frac{\partial^2 F_1}{\partial q_s \partial q_\ell} & \frac{\partial q_\ell}{\partial G} \end{vmatrix} + \begin{vmatrix} \frac{\partial q_s}{\partial F} & \frac{\partial^2 F_1}{\partial q_s \partial Q_\ell} & \frac{\partial Q_\ell}{\partial F} \\ \frac{\partial q_s}{\partial G} & \frac{\partial^2 F_1}{\partial q_s \partial Q_\ell} & \frac{\partial Q_\ell}{\partial G} \end{vmatrix}$$

$$-\frac{\partial^2 F_1}{\partial q_s \partial q_\ell} = \begin{vmatrix} \frac{\partial q_s}{\partial F} & \frac{\partial q_\ell}{\partial F} \\ \frac{\partial q_s}{\partial G} & \frac{\partial q_\ell}{\partial G} \end{vmatrix} + \frac{\partial^2 F_1}{\partial q_s \partial Q_\ell} \begin{vmatrix} \frac{\partial q_s}{\partial F} & \frac{\partial Q_\ell}{\partial F} \\ \frac{\partial q_s}{\partial G} & \frac{\partial Q_\ell}{\partial G} \end{vmatrix}$$

where $\ell, s = 1, 2, 3, \ldots n$ are dummy indices so that the 1st determinant on R.H.S is zero and we are left with

$$\{F, G\}_{q,p} = \frac{\partial^2 F_1}{\partial q_s \partial Q_\ell} \begin{vmatrix} \frac{\partial q_s}{\partial F} & \frac{\partial Q_\ell}{\partial F} \\ \frac{\partial q_s}{\partial G} & \frac{\partial Q_\ell}{\partial G} \end{vmatrix} \quad \text{——— (9)}$$

Replace q by Q in (9), we get

$$\{F, G\}_{Q,P} = 0 = \frac{\partial^2 F_1}{\partial Q_s \partial Q_\ell} \begin{vmatrix} \frac{\partial Q_s}{\partial F} & \frac{\partial Q_\ell}{\partial F} \\ \frac{\partial Q_s}{\partial G} & \frac{\partial Q_\ell}{\partial G} \end{vmatrix} \quad \text{———}$$

Adding the corresponding sides of (9) and (10),

$$\{F, G\}_{q,p} + 0 = \frac{\partial^2 F_1}{\partial q_s \partial Q_\ell} \begin{vmatrix} \frac{\partial q_s}{\partial F} + \frac{\partial Q_s}{\partial F} & \frac{\partial Q_\ell}{\partial F} \\ \frac{\partial q_s}{\partial G} + \frac{\partial Q_s}{\partial G} & \frac{\partial Q_\ell}{\partial G} \end{vmatrix}$$

$$\{F, G\}_{q,p} = \begin{vmatrix} \frac{\partial^2 F_1}{\partial q_s \partial Q_\ell} \frac{\partial q_s}{\partial F} + \frac{\partial^2 F_1}{\partial q_s \partial Q_\ell} \frac{\partial Q_s}{\partial F} & \frac{\partial Q_\ell}{\partial F} \\ \frac{\partial^2 F_1}{\partial q_s \partial Q_\ell} \frac{\partial q_s}{\partial G} + \frac{\partial^2 F_1}{\partial q_s \partial Q_\ell} \frac{\partial Q_s}{\partial G} & \frac{\partial Q_\ell}{\partial G} \end{vmatrix}$$

Note that

$$\frac{\partial}{\partial F}\left(\frac{\partial F_1}{\partial Q_i}\right) = \frac{\partial^2 F_1}{\partial q_s \partial Q_i}\frac{\partial q_s}{\partial F} + \frac{\partial^2 F_1}{\partial q_s \partial Q_i}\frac{\partial Q_s}{\partial F}$$

and $$\frac{\partial}{\partial G}\left(\frac{\partial F_1}{\partial Q_i}\right) = \frac{\partial^2 F_1}{\partial q_s \partial Q_i}\frac{\partial q_s}{\partial G} + \frac{\partial^2 F_1}{\partial q_s \partial Q_i}\frac{\partial Q_s}{\partial G}$$

(11) $\Rightarrow \{F, G\}_{q,p} = \begin{vmatrix} \frac{\partial}{\partial F}\left(\frac{\partial F_1}{\partial Q_i}\right) & \frac{\partial Q_i}{\partial F} \\ \frac{\partial}{\partial G}\left(\frac{\partial F_1}{\partial Q_i}\right) & \frac{\partial Q_i}{\partial G} \end{vmatrix}$ ——— (12)

But for $s=i$ the relation (3) implies that

$$P_i = -\frac{\partial F_1}{\partial Q_i} \Rightarrow \frac{\partial F_1}{\partial Q_i} = -P_i$$

$$\Rightarrow \frac{\partial}{\partial F}\left(\frac{\partial F_1}{\partial Q_i}\right) = -\frac{\partial P_i}{\partial F} \text{ and } \frac{\partial}{\partial G}\left(\frac{\partial F_1}{\partial Q_i}\right) = -\frac{\partial P_i}{\partial G}$$

(12) $\Rightarrow \{F, G\}_{q,p} = \begin{vmatrix} -\frac{\partial P_i}{\partial F} & \frac{\partial Q_i}{\partial F} \\ -\frac{\partial P_i}{\partial G} & \frac{\partial Q_i}{\partial G} \end{vmatrix}$

$$= -\begin{vmatrix} \frac{\partial P_i}{\partial F} & \frac{\partial Q_i}{\partial F} \\ \frac{\partial P_i}{\partial G} & \frac{\partial Q_i}{\partial G} \end{vmatrix}$$

$$= -\begin{vmatrix} \frac{\partial Q_i}{\partial F} & \frac{\partial P_i}{\partial F} \\ \frac{\partial Q_i}{\partial G} & \frac{\partial P_i}{\partial G} \end{vmatrix} \quad \text{by } C_{12}$$

$$= \sum_{i=1}^{n}\left(\frac{\partial Q_i}{\partial F}\frac{\partial P_i}{\partial G} - \frac{\partial Q_i}{\partial G}\frac{\partial P_i}{\partial F}\right) = \{F, G\}_{q,p}$$

Hence the Lagrange bracket is invariant under canonical transformation.

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Question:-

Establish the relationship between Lagrange's and Poisson brackets.

OR Show that Lagrange and poisson brackets are reciprocal to each other.

OR Show that $\sum_{i=1}^{2n} \{F_i, F_j\}[F_i, F_k] = \delta_{jk}$

Solution.

Let us suppose that $q_1, q_2, \cdots q_n, p_1, p_2, \cdots p_n$ be 2n independent functions, in a domain of phase space, of the 2n arguments $F_1, F_2, \cdots F_n, F_{n+1}, \cdots F_{2n}$ the transformation is such that it gives a topological mapping of the domain of the phase space on a domain of F-space.

We want to show that

$$\sum_{i=1}^{2n} \{F_i, F_j\}[F_i, F_k] = \delta_{jk} \quad \text{——— (1)}$$

By Lagrange and poisson brackets, we can write

$$\{F_i, F_j\} = \sum_{s=1}^{n} \left(\frac{\partial q_s}{\partial F_i} \frac{\partial p_s}{\partial F_j} - \frac{\partial p_s}{\partial F_i} \frac{\partial q_s}{\partial F_j} \right)$$

and $$[F_i, F_k] = \sum_{s=1}^{n} \left(\frac{\partial F_i}{\partial q_s} \frac{\partial F_k}{\partial p_s} - \frac{\partial F_i}{\partial p_s} \frac{\partial F_k}{\partial q_s} \right)$$

By First Summation convention,

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$$\{F_i, F_j\} = \frac{\partial q_s}{\partial F_i}\frac{\partial p_s}{\partial F_j} - \frac{\partial p_s}{\partial F_i}\frac{\partial q_s}{\partial F_j} \quad \text{---(2)}$$

$$[F_i, F_k] = \frac{\partial F_i}{\partial q_r}\frac{\partial F_k}{\partial p_r} - \frac{\partial F_i}{\partial p_r}\frac{\partial F_k}{\partial q_r} \quad \text{---(3)}$$

By (2) and (3), we have,

$$\{F_i, F_j\}[F_i, F_k] = \frac{\partial q_s}{\partial F_i}\frac{\partial p_s}{\partial F_j}\frac{\partial F_i}{\partial q_r}\frac{\partial F_k}{\partial p_r} -$$

$$\frac{\partial q_s}{\partial F_i}\frac{\partial p_s}{\partial F_j}\frac{\partial F_i}{\partial p_r}\frac{\partial F_k}{\partial q_r} - \frac{\partial p_s}{\partial F_i}\frac{\partial q_s}{\partial F_j}\frac{\partial F_i}{\partial q_r}\frac{\partial F_k}{\partial p_r}$$

$$+ \frac{\partial p_s}{\partial F_i}\frac{\partial q_s}{\partial F_j}\frac{\partial F_i}{\partial p_r}\frac{\partial F_k}{\partial q_r} \quad \text{---(4)}$$

Note that

$$\frac{\partial q_s}{\partial F_i}\frac{\partial F_i}{\partial q_r} = \frac{\partial q_s}{\partial q_r} = \delta_{sr} \; ; \; \frac{\partial p_s}{\partial F_i}\frac{\partial F_i}{\partial q_r} = \frac{\partial p_s}{\partial q_r} =$$

$$\frac{\partial q_s}{\partial F_i}\frac{\partial F_i}{\partial p_r} = \frac{\partial q_s}{\partial p_r} = 0 \; ; \; \frac{\partial p_s}{\partial F_i}\frac{\partial F_i}{\partial p_r} = \frac{\partial p_s}{\partial p_r} =$$

Thus equation (4) becomes

$$\{F_i, F_j\}[F_i, F_k] = \frac{\partial q_s}{\partial F_i}\frac{\partial F_i}{\partial q_r}\frac{\partial p_s}{\partial F_j}\frac{\partial F_k}{\partial p_r} - \cancel{\frac{\partial q_s}{\partial F_i}\frac{\partial F_i}{\partial p_r}\frac{\partial p_s}{\partial F_j}}^{0}$$

$$\cancel{-\frac{\partial p_s}{\partial F_i}\frac{\partial F_i}{\partial q_r}\frac{\partial q_s}{\partial F_j}\frac{\partial F_k}{\partial p_r}}^{0} + \frac{\partial p_s}{\partial F_i}\frac{\partial F_i}{\partial p_r}\frac{\partial q_s}{\partial F_j}\frac{\partial F_k}{\partial q_r}$$

$$= \delta_{sr}\frac{\partial p_s}{\partial F_j}\frac{\partial F_k}{\partial p_r} + \delta_{sr}\frac{\partial q_s}{\partial F_j}\frac{\partial F_k}{\partial q_r}$$

$$\Rightarrow \{F_i, F_j\}[F_i, F_k] = \begin{cases} \dfrac{\partial p_s}{\partial F_j}\dfrac{\partial F_k}{\partial p_r} + \dfrac{\partial q_s}{\partial F_j}\dfrac{\partial F_k}{\partial q_r} & \text{if } s = r \\ 0 & \text{if } s \neq r \end{cases}$$

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$$\Rightarrow \{F_i, F_j\}[F_i, F_k] = \frac{\partial p_s}{\partial F_j}\frac{\partial F_k}{\partial p_s} + \frac{\partial q_s}{\partial F_j}\frac{\partial F_k}{\partial q_s}, \quad s=$$

$$= \frac{\partial F_k}{\partial p_s}\frac{\partial p_s}{\partial F_j} + \frac{\partial F_k}{\partial q_s}\frac{\partial q_s}{\partial F_j}$$

$$= \frac{\partial F_k}{\partial F_j}(p_s, q_s) = \delta_{jk}$$

~~Now equation (1) becomes~~

$$\Rightarrow \sum_{i=1}^{n} \{F_i, F_j\}[F_i, F_k] = \delta_{jk}$$ which is required result.

Question:-

If the Hamiltonian of a system is $H = \frac{p^2}{2} - \frac{1}{2q^2}$ then show that $F = \frac{pq}{2} - Ht$ is a constant of motion.

Solution.

Given that $F = \frac{pq}{2} - Ht$

and $H = \frac{p^2}{2} - \frac{1}{2q^2}$

Therefore $F = \frac{pq}{2} - \left(\frac{p^2}{2} - \frac{1}{2q^2}\right)t$

Since we know that equation of motion in Poisson bracket form is

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H]$$

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Also we know that the condition of constant of motion is

$$\frac{dF}{dt} = 0$$

or $$\frac{\partial F}{\partial t} + [F, H] = 0 \quad \text{——— (1)}$$

$$\frac{\partial F}{\partial t} = -\left(\frac{p^2}{2} - \frac{1}{2q^2}\right) = -\frac{p^2}{2} + \frac{1}{2q^2} \quad \text{——— (2)}$$

and $$[F, H] = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} \quad \text{——— (3)}$$

$$\frac{\partial F}{\partial q} = \frac{p}{2} - \frac{1}{q^3}t \quad ; \quad \frac{\partial F}{\partial p} = \frac{q}{2} - pt$$

$$\frac{\partial H}{\partial q} = \frac{1}{q^3} \quad ; \quad \frac{\partial H}{\partial p} = p$$

So (3) $\Rightarrow [F, H] = \left(\frac{p}{2} - \frac{1}{q^3}t\right)p - \left(\frac{q}{2} - pt\right)\frac{1}{q^3}$

$$= \frac{p^2}{2} - \frac{pt}{q^3} - \frac{1}{2q^2} + \frac{pt}{q^3}$$

$$\Rightarrow [F, H] = \frac{p^2}{2} - \frac{1}{2q^2} \quad \text{——— (4)}$$

Adding (2) and (4), we get

$$\frac{\partial F}{\partial t} + [F, H] = -\frac{p^2}{2} + \frac{1}{2q^2} + \frac{p^2}{2} - \frac{1}{2q^2}$$

$$= 0$$

Hence (1) is satisfied so F is a consta of motion.

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Question. :-
A particle of mass m moves in a force fie
of potential V. Write Hamilton's equation
in spherical coordinates (r, θ, φ).
Solution.
The K.E. in spherical coordinates (r, θ, φ) is
given as

$$K.E = T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta)$$

The Lagrangian function is

$$L = T - V$$

$$\Rightarrow L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) - V(r, \theta, \varphi)$$

where $V(r, \theta, \varphi)$ is P.E in spherical coordinates (r, θ, φ).
The Hamiltonian is given by

$$H = T + V$$

$$\Rightarrow H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) + V(r, \theta, \varphi)$$

From equation (1), we can write

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m(2r^2\dot{\theta}) = mr^2\dot{\theta}$$

$$\Rightarrow p_\theta = mr^2\dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2} \quad \text{---------- (3)}$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{1}{2}m(2\dot{r}) = m\dot{r}$$

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$\Rightarrow p_r = m\dot{r} \Rightarrow \dot{r} = p_r/m$ ——— (4)

and

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2} m(2r^2 \sin^2\theta \dot{\phi}) = mr^2 \sin^2\theta \dot{\phi}$$

$$\Rightarrow p_\phi = mr^2 \sin^2\theta \dot{\phi}$$

$$\Rightarrow \dot{\phi} = \frac{p_\phi}{mr^2 \sin^2\theta} \quad \text{——— (5)}$$

Subsituting (3), (4) and (5) in equation (2), we get

$$H = \frac{1}{2} m \left[\frac{p_r^2}{m^2} + r^2 \frac{p_\theta^2}{m^2 r^4} + r^2 \sin^2\theta \frac{p_\phi^2}{m^2 r^4 \sin^4\theta} \right] + V(r,\theta,\phi)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} + V(r,\theta,\phi). \quad \text{——— (6)}$$

Now we find the Hamilton's equations of motion by using

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \quad \text{and} \quad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$$

For $\alpha = 1, 2, 3$.

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} \; ; \; \dot{q}_1 = \frac{\partial H}{\partial p_1}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} \; ; \; \dot{q}_2 = \frac{\partial H}{\partial p_2}$$

and $\dot{p}_3 = -\frac{\partial H}{\partial q_3} \; ; \; \dot{q}_3 = \frac{\partial H}{\partial p_3}$

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$$\Rightarrow \dot{p}_s = \frac{-\partial H}{\partial s} \quad ; \dot{s} = \dot{q}_1 = \frac{\partial H}{\partial p_s} \quad \text{---(7)}$$

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} \quad ; \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} \quad \text{---(8)}$$

$$\dot{p}_\varphi = \frac{-\partial H}{\partial \varphi} \quad ; \quad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi} \quad \text{---(9)}$$

From equations (6), (7), (8) and (9), we have.

$$\dot{p}_s = \frac{p_\theta^2}{ms^3} + \frac{p_\varphi^2}{ms^3 \sin^2\theta} - \frac{\partial V}{\partial s}$$

$$\dot{s} = \frac{p_s}{m} \quad ; \quad \dot{\theta} = \frac{p_\theta}{ms^2} \quad ; \quad \dot{\varphi} = \frac{p_\varphi}{ms^2 \sin^2\theta}$$

$$\dot{p}_\theta = \frac{p_\varphi^2 \cos\theta}{ms^2 \sin^3\theta} - \frac{\partial V}{\partial \theta}$$

$$\dot{p}_\varphi = \frac{-\partial V}{\partial \varphi} \quad \text{as required.}$$

Hamilton's Jacobi Equation:-

Let us consider a dynamical system with Hamiltonian $H(t, q_s, p_s)$. The equations of the generating function $S(t, q_s, p_s)$ are

$$p_s = \frac{\partial S}{\partial q_s}, \quad Q_s = \frac{\partial S}{\partial p_s} \quad \text{and} \quad K = H + \frac{\partial}{}$$

where K is our new Hamiltonian.

The equations for canonical transformation are

$$\dot{Q}_s = \frac{\partial K}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial K}{\partial q_s}$$

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want to find the transformation for which new Hamiltonian $K=0$.

$$\Rightarrow \frac{\partial S}{\partial t} + H(t, q_s, p_s) = 0$$

$$\Rightarrow \frac{\partial S}{\partial t} + H\left(t, q_s, \frac{\partial S}{\partial q_s}\right) = 0 \quad \text{---} (3)$$

$\Rightarrow \dot{Q}_s = 0$ and $\dot{P}_s = 0$ $\because K = 0$.

$\Rightarrow Q_s = \text{Const}^{tt}$ and $P_s = \text{Const}^{tt}$

$Q_s = \alpha_s$ (say) and $P_s = \beta_s$ (say).

$\Rightarrow q_s = q_s(t)$; $p_s = p_s(t)$

Example 1. 2010 2012, 2015, 2019 :- ☆

Use the Hamilton-Jacobi method to obtain the motion of one dimensional harmonic oscillator of mass m.

Solution.

Let q be the position coordinate of one dimensional harmonic oscillator. Then $\dot{q}$ be its velocity.

The K.E. for one dimensional harmonic oscillator is $T = \frac{1}{2} m \dot{q}^2$

and the P.E is

$$V = \frac{1}{2} k q^2$$

The Lagrangian for the system is

$$L = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2 \quad \text{---} (1)$$

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The Hamiltonian for the system is

$$H = T + V = \frac{1}{2} m\dot{q}^2 + \frac{1}{2} Kq^2 \quad \text{——— (2)}$$

We want to eliminate $\dot{q}$ from equation

So (1) implies

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \Rightarrow p = m\dot{q}$$

$$\Rightarrow \dot{q} = p/m$$

Thus equation (2) becomes

$$H = \frac{1}{2} m \frac{p^2}{m^2} + \frac{1}{2} Kq^2$$

$$\Rightarrow H = \frac{1}{2} p^2/m + 1/2\, Kq^2 \quad \text{——— (3)}$$

Now we write Hamilton Jacobi equation as follows

$$\frac{\partial S}{\partial t} + H\left(t, q_s, \frac{\partial S}{\partial q_s}\right) = 0$$

$$\Rightarrow \frac{\partial S}{\partial t} + H\left(t, q, \frac{\partial S}{\partial q}\right) = 0 \quad \because \text{for } s = 1.$$

$$\Rightarrow \frac{\partial S}{\partial t} + \frac{1}{2}\frac{p^2}{m} + \frac{1}{2} Kq^2 = 0$$

$$\Rightarrow \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} Kq^2 = 0 \quad \because p = \frac{\partial S}{\partial q} \longrightarrow (4)$$

Suppose $S = S_1(q) + S_2(t)$ is solution of (4)

Then $$\frac{\partial S}{\partial t} = \frac{\partial S_2}{\partial t} = \frac{dS_2}{dt}$$

and $$\frac{\partial S}{\partial q} = \frac{\partial S_1}{\partial q} = \frac{dS_1}{dq}$$

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Now equation (4) becomes

$$\frac{dS_2}{dt} + \frac{1}{2m}\left(\frac{dS_1}{dq_1}\right)^2 + \frac{1}{2}Kq_1^2 = 0$$

$$\Rightarrow \frac{1}{2m}\left(\frac{dS_1}{dq_1}\right)^2 + \frac{1}{2}Kq_1^2 = -\frac{dS_2}{dt} \quad \text{---(5)}$$

Since the L.H.S of (5) is function of q only and the R.H.S is function of t only, both sides are equal only if they are equal to some constant say β.

$$\Rightarrow \frac{1}{2m}\left(\frac{dS_1}{dq_1}\right)^2 + \frac{1}{2}Kq_1^2 = \beta \quad \text{---(6)} \quad ; \quad -\frac{dS_2}{dt} = \beta \quad \text{---(7)}$$

(7) $\Rightarrow -S_2 = \beta t + C$

where C is an arbitrary constant of integration

$$S_2 = -\beta t \quad \text{---(8) (ignoring constant)}$$

(6) $\Rightarrow \left(\frac{dS_1}{dq_1}\right)^2 + mKq_1^2 = 2m\beta$

$$\Rightarrow \left(\frac{dS_1}{dq_1}\right)^2 = 2m\beta - mKq_1^2$$

$$\Rightarrow \frac{dS_1}{dq_1} = \sqrt{2m\beta - mKq_1^2}$$

Integrating, we get

$$\Rightarrow S_1 = \int \sqrt{2m\beta - mKq_1^2}\, dq_1$$

Thus $S = S_1(q_1) + S_2(t) = \int \sqrt{2m\beta - mKq_1^2}\, dq_1 - \beta t$

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The Legendre equation for ~~canonical~~ generating function $S(t, q, P)$ will give

$$Q = \frac{\partial S}{\partial P} = \frac{\partial S}{\partial \beta} \quad \because P = \text{const}^{tt} = \beta \text{ (say)}$$

$$\Rightarrow \gamma = \frac{\partial S}{\partial \beta} \quad \because Q = \text{const}^{tt} = \gamma \text{ (say)}$$

$$\Rightarrow \gamma = \frac{\partial}{\partial \beta} \int \sqrt{2m\beta - mkq^2}\, dq - \frac{\partial}{\partial \beta} \beta t$$

$$= \int \frac{\partial}{\partial \beta} \sqrt{2m\beta - mkq^2}\, dq - \beta t$$

$$= \int \left\{ \frac{1}{2} (2m\beta - mkq^2)^{-1/2} (2m) \right\} dq - \beta t$$

$$= \int \frac{m}{\sqrt{2m\beta - mkq^2}}\, dq - \beta t$$

$$= \frac{m}{\sqrt{mk}} \int \frac{dq}{\sqrt{\frac{2\beta}{k} - q^2}} - \beta t$$

$$\gamma = \sqrt{\frac{m}{k}} \int \frac{1}{\sqrt{\left(\sqrt{\frac{2\beta}{k}}\right)^2 - (q)^2}}\, dq - \beta t \quad \text{---(8)}$$

Since $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right)$

Therefore

$$(8) \Rightarrow \gamma = \sqrt{\frac{m}{k}} \sin^{-1}\left(\frac{q}{\sqrt{\frac{2\beta}{k}}}\right) - \beta t \quad \because a = \sqrt{2\beta/k},\ x = q$$

$$\Rightarrow \sqrt{\frac{m}{k}} \sin^{-1}\left(q\sqrt{k/2\beta}\right) = t + \gamma$$

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$\Rightarrow \sin^{-1}\sqrt{K/2\beta} = \sqrt{\frac{K}{m}}(t+\gamma)$

$\Rightarrow q\sqrt{K/2\beta} = \sin\sqrt{K/m}(t+\gamma)$

$\Rightarrow q = \sqrt{\frac{2\beta}{K}}\sin\sqrt{\frac{K}{m}}(t+\gamma)$ ——— (1)

which is required solution.

Example 2. :—

Apply the Hamilton-Jacobi method to study the motion of freely falling body.

Solution.

Let us consider z-axis as vertical direction Then the K.E. and P.E of the system are given by

$\bar{T} = \frac{1}{2}m\dot{z}^2$; $V = mgz$

The Lagrangian is

$L = \bar{T} - V$

$\Rightarrow L = \frac{1}{2}m\dot{z}^2 - mgz.$ ——— (1)

The Hamiltonian is

$H = \bar{T} + V = \frac{1}{2}m\dot{z}^2 + mgz$ ——— (2)

By (1) $\frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow p = m\dot{z}$

$\Rightarrow \dot{z} = p/m$

(2) $\Rightarrow H = \frac{1}{2}m\frac{p^2}{m^2} + mgz$

$\Rightarrow H = \frac{p^2}{2m} + mgz$ ——— (3)

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Now we write the Hamilton Jacobi equati
as follows.

$$\frac{\partial S}{\partial t} + H\left(t, q_s, \frac{\partial S}{\partial q_s}\right) = 0$$

$$\Rightarrow \frac{\partial S}{\partial t} + \frac{p^2}{2m} + mgz = 0.$$

$$\Rightarrow \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial z}\right)^2 + mgz = 0 \quad \text{---} (4)$$

$\because p =$

Suppose $S = S_1(z) + S_2(t)$ is solution of

Then $\frac{\partial S}{\partial z} = \frac{\partial S_1}{\partial z} = \frac{dS_1}{dz}$

$$\frac{\partial S}{\partial t} = \frac{\partial S_2}{\partial t} = \frac{dS_2}{dt}$$

Now equation (4) becomes

$$\frac{dS_2}{dt} + \frac{1}{2m}\left(\frac{dS_1}{dz}\right)^2 + mgz = 0$$

$$\Rightarrow \frac{1}{2m}\left(\frac{dS_1}{dz}\right)^2 + mgz = -\frac{dS_2}{dt} \quad \text{---} (5)$$

Since the L.H.S of (5) is function of z o
and the R.H.S is function of t only, both
equal only if they are equal to some constan
say β.

$$(5) \Rightarrow \frac{1}{2m}\left(\frac{dS_1}{dz}\right)^2 + mgz = \beta \quad \text{---} (6)$$

$$-\frac{dS_2}{dt} = \beta \quad \text{---} (7)$$

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$\Rightarrow S_2 = -\beta t$ ——— (2)

$\Rightarrow \left(\frac{dS_1}{dz}\right)^2 + 2m^2gz = 2m\beta$

$\Rightarrow \left(\frac{dS_1}{dz}\right)^2 = 2m\beta - 2m^2gz$

$\frac{dS_1}{dz} = \sqrt{2m(\beta - mgz)}$

Integrating, we get

$S_1 = \int \sqrt{2m(\beta - mgz)}\, dz$

and

$S = \int \sqrt{2m(\beta - mgz)}\, dz - \beta t$

The Legendre equation for generating function S are.

$Q = \frac{\partial S}{\partial \beta} \Rightarrow \gamma = \frac{\partial S}{\partial \beta}$ (say).

$\Rightarrow \gamma = \frac{\partial}{\partial \beta}\int \sqrt{2m(\beta - mgz)}\, dz - \frac{\partial \beta t}{\partial \beta}$

$= \int \frac{\partial}{\partial \beta} \sqrt{2m(\beta - mgz)}\, dz - t$

$\Rightarrow \gamma + t = \int \sqrt{2m}\, \frac{\partial}{\partial \beta} \sqrt{\beta - mgz}\, dz.$

$\Rightarrow \gamma + t = \sqrt{2m} \int \frac{1}{2}(\beta - mgz)^{-1/2}(1)\, dz.$

$= -\frac{\sqrt{2m}}{2mg} \int (\beta - mgz)^{-1/2}(-mg)\, dz$ {Xing and Dividing by $-mg$.}

$= -\frac{\sqrt{2m}}{2mg}\left(\beta - mgz\right)^{1/2}$

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$$\Rightarrow \gamma + t = -\sqrt{\frac{2}{m}}\,\frac{\sqrt{\beta - mgz}}{g}$$

$$\Rightarrow \sqrt{\frac{m}{2}}\,(\gamma+t)g = -\sqrt{\beta - mgz}$$

Taking square of both sides, we get

$$(\beta - mgz) = \left[\sqrt{\frac{m}{2}}\,(\gamma+t)g\right]^2$$

$$\Rightarrow -mgz = \frac{m}{2}(\gamma+t)^2 g^2 - \beta$$

$$\Rightarrow z = \frac{\beta}{mg} - \frac{1}{2}(\gamma+t)^2 g \qquad \text{---(9)}$$

which is required solution.

Example 3. 2012;

Use Hamilton-Jacobi method to solve Kepler's problem for a particle in an inverse square central force field.

Solution.

The K.E. and P.E. of the system are

$$K.E = \bar{T} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad ; \quad V = -K/r$$

The Lagrangian is

$$L = \bar{T} - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + K/r \qquad \text{---(1)}$$

The Hamiltonian is

$$H = \bar{T} + V$$

$$\Rightarrow H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - K/r \qquad \text{---(2)}$$

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① $\frac{\partial L}{\partial \dot{r}} = m.\dot{r} \Rightarrow p_r = m.\dot{r} \Rightarrow \dot{r} = p_r/m$

and $\frac{\partial L}{\partial \dot{\theta}} = m.r^2\dot{\theta} \Rightarrow p_\theta = m.r^2\dot{\theta}$

$\Rightarrow \dot{\theta} = \frac{p_\theta}{m.r^2}$

$\Rightarrow H = \frac{1}{2m}\left(p_r^2 + \dot{r}^2 \frac{p_\theta^2}{m^2.r^4}\right) - \frac{K}{r}$

$\Rightarrow H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2m.r^2} - \frac{K}{r}$ ——— (3)

The Hamilton Jacobi equation is

$\frac{\partial S}{\partial t} + H\left(t, q_s, \frac{\partial S}{\partial q_s}\right) = 0$

$\Rightarrow \frac{\partial S}{\partial t} + \frac{p_r^2}{2m} + \frac{p_\theta^2}{2m.r^2} - \frac{K}{r} = 0$ ——— (4)

Suppose $S = S_1(r) + S_2(\theta) + S_3(t)$ is solution of equation (4).

Then $\frac{\partial S}{\partial r} = \frac{\partial S_1}{\partial r} = \frac{dS_1}{dr}$

$\frac{\partial S}{\partial \theta} = \frac{\partial S_2}{\partial \theta} = \frac{dS_2}{d\theta}$

and $\frac{\partial S}{\partial t} = \frac{\partial S_3}{\partial t} = \frac{dS_3}{dt}$

Now equation (4) becomes

$\frac{dS_3}{dt} + \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2m.r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - \frac{K}{r} = 0.$

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$$\Rightarrow \frac{dS_3}{dt} + \frac{1}{2m}\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS_2}{d\theta}\right)^2 - \frac{K}{r} = 0$$

$$\Rightarrow \frac{1}{2m}\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS_2}{d\theta}\right)^2 - \frac{K}{r} = -\frac{dS_3}{dt} \quad \text{—— (5)}$$

Since the L.H.S of equation (5) is function of r, θ and R.H.S of (5) is function of t only; both sides are equal only if they are equal to some constant say β.

$$(5) \Rightarrow \frac{1}{2m}\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS_2}{d\theta}\right)^2 - \frac{K}{r} = \beta \quad \text{——}$$

and $$-\frac{dS_3}{dt} = \beta \quad \text{—— (7)}$$

Solving (7), we get

$$S_3 = -\beta t \quad \text{—— (8)}$$

$$(6) \Rightarrow r^2\left(\frac{dS_1}{dr}\right)^2 + \left(\frac{dS_2}{d\theta}\right)^2 - 2mKr = 2m\beta r^2$$

$$\Rightarrow \left(\frac{dS_2}{d\theta}\right)^2 = 2m\beta r^2 - r^2\left(\frac{dS_1}{dr}\right)^2 + 2mKr$$

Since the L.H.S of (9) is function of θ and the R.H.S is function of r only. Sides are equal only if they are equal Some constant say β_1.

$$(9) \Rightarrow \left(\frac{dS_2}{d\theta}\right)^2 = \beta_1 \quad \text{—— (10)}$$

and $$2m\beta r^2 - r^2\left(\frac{dS_1}{dr}\right)^2 + 2mKr = \beta_1$$

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$\Rightarrow \left(\frac{dS_2}{d\theta}\right) = \beta_1^{\frac{1}{2}} \Rightarrow S_2 = \beta_1^{\frac{1}{2}} \int d\theta$

$\Rightarrow S_2 = \beta_1^{\frac{1}{2}} \theta = \beta_2 \theta$ say $\beta_2 = \beta_1^{\frac{1}{2}}$

$\Rightarrow S_2 = \beta_2 \theta$ ——— (12)

(11) $\Rightarrow -\dot{s}^2 \left(\frac{dS_1}{ds}\right)^2 = \beta_1 - 2m\beta\dot{s}^2 - 2mK\dot{s}$

$\Rightarrow \left(\frac{dS_1}{ds}\right)^2 = -\frac{\beta_1}{\dot{s}^2} + 2m\beta + \frac{2mK}{\dot{s}}$

$\Rightarrow \frac{dS_1}{ds} = \sqrt{2m\beta - \beta_1/\dot{s}^2 + \frac{2mK}{\dot{s}}}$

Integrating, we get,

$S_1 = \int\int 2m\beta - \frac{\beta_1}{\dot{s}^2} + \frac{2mK}{\dot{s}}\, ds$ ——— (13)

Subsituting (8), (12) and (13) in S, we get.

$S = \int\int 2m\beta - \frac{\beta_1}{\dot{s}^2} + \frac{2Km}{\dot{s}}\, ds + \beta_2\theta - \beta t$

The Legendre equation of generating function S are

$Q_1 = Q_2 = \frac{\partial S}{\partial \beta_1} \Rightarrow \gamma_1 = \frac{\partial S}{\partial \beta}$ (say).

$\Rightarrow \gamma_1 = \frac{\partial}{\partial \beta}\int\int 2m\beta - \frac{\beta_1}{\dot{s}^2} + \frac{2mK}{\dot{s}}\, ds + \frac{\partial}{\partial \beta}\beta_2\theta - \frac{\partial \beta t}{\partial \beta}$

$\Rightarrow \gamma_1 = \int \frac{\partial}{\partial \beta}\left[2m\beta - \frac{\beta_1}{\dot{s}^2} + \frac{2mK}{\dot{s}}\right] ds - t$

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$$\Rightarrow \gamma_1 + t = \int \frac{1}{2}\left(2m\beta - \frac{\beta_1}{r^2} + \frac{2mK}{r}\right)^{-1/2} (2m)\, dr$$

$$\Rightarrow \int \frac{m\, dr}{\sqrt{2m\beta - \frac{\beta_1}{r^2} + \frac{2mK}{r}}} = \gamma_1 + t \quad \text{---(14)}$$

and $Q_2 = \frac{\partial S}{\partial \beta_2} \Rightarrow \gamma_2 = \frac{\partial S}{\partial \beta_2}$ (say)

$$\Rightarrow \gamma_2 = \frac{\partial}{\partial \beta_2} \int \sqrt{2m\beta - \frac{\beta_2^2}{r^2} + \frac{2mK}{r}}\, dr + \frac{\partial \beta_2}{\partial \beta_2}\theta - \frac{\partial}{\partial}$$

$$\Rightarrow \gamma_2 = \int \frac{\partial}{\partial \beta_2} \sqrt{2m\beta - \frac{\beta_2^2}{r^2} + \frac{2mK}{r}}\, dr + \theta$$

$$\Rightarrow \gamma_2 - \theta = \int \frac{1}{2}\left(2m\beta - \beta_2^2/r^2 + \frac{2mK}{r}\right)^{-1/2} \left(-\frac{2\beta_2}{r^2}\right) dr$$

$$\Rightarrow \int \frac{\beta_2\, dr}{r^2\sqrt{2m\beta - \frac{\beta_2^2}{r^2} + \frac{2mK}{r}}} \quad \theta = \theta - \gamma_2 \quad \text{---(15)}$$

After solving the integrals in equations (14) and (15) we can find r and θ.

Example 4: 2016, 2019/A :—

Use Hamilton Jacobi method to obtain the equations of motion for a particle of mass m moving in a force field whose potential energy in Spherical Coordinates (r, θ, φ) is $V = -\frac{K\cos\theta}{r^2}$.

(OR)

A particle of mass m moves in a force field of potential V if $V = -\frac{K\cos\theta}{r^2}$ write down Hamilton Jacobi equation.

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esponding to the cyclic coordinates is given

$$\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}^{0} = 0 \qquad \because L \neq L(q_i).$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\frac{\partial L}{\partial \dot{q}_i} = \text{constant}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \beta_i \qquad \because \beta_i = \text{constant}$$

other words if p_i = constant then the rresponding generalized coordinates are cyclic ignorable coordinates.

he Hamilton's equations of motion are

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \text{and} \quad \frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$\Rightarrow \frac{\partial H}{\partial q_i} = 0 \qquad \because \dot{p}_i = 0 \text{ for cyclic coordi..}$$

outhian Equation of motion. :—

rove that $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) - \frac{\partial R}{\partial q_j} = 0$ 2008, 2009, 2012.

where $j = m+1, m+2, \ldots, n$. Here first m-coordinates are cyclic or ignorable coordinates and $R = L - \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i = L - \sum_{i=1}^{m} \beta_i \dot{q}_i$

$\because \beta_i = \partial L / \partial \dot{q}_i$ is..

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is called Routhian function.

Proof.

We shall prove that R plays the role of Lagra function for a dynamical system with the Coordinates $q_{m+1}, q_{m+2}, \cdots, q_n$. This metho actually reduces the original problem involv n-degrees of freedom to a problem of a system with n-m degrees of freedom. This process Known as ignoration of the coordinates. These Coordinates appear in the system containing gyroscopes and usually such systems are called gyroscopic systems.

The Lagrange's equations of motion for a system having n-degrees of freedom is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \longrightarrow (1)$$

where $s = 1, 2, \cdots m, m+1, m+2, \cdots n$.

Let us assume that first m- coordinates are cyclic so that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}^{\nearrow 0} = 0 \qquad \because L \neq LC$$

$$\Rightarrow \frac{d}{dt}(p_i) = 0$$

$$\Rightarrow p_i = \frac{\partial L}{\partial \dot{q}_i} = \text{Constant}$$

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Integrating by parts, we get

$$I_1 = p_s \delta q_s \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} (\dot{p}_s \delta q_s)\, dt$$

$$\Rightarrow I_1 = p_s \delta q_s - p_s^* \delta q_s^* + \int_{t_1}^{t_2} \left(\frac{\partial H}{\partial q_s} \delta q_s\right) dt$$

where staric denotes the value which has been taken at initial time.

Thus relation (2) gives

$$\delta S = p_s \delta q_s - p_s^* \delta q_s^* + \int_{t_1}^{t_2} \left(\frac{\partial H}{\partial q_s} \delta q_s\right) dt - \int_{t_1}^{t_2} \left(\frac{\partial H}{\partial q_s} \delta q_s\right) dt$$

$$\Rightarrow \delta S = p_s \delta q_s - p_s^* \delta q_s^* \longrightarrow (3)$$

$\Rightarrow \delta S$ contains the differential of q_s and q_s^*.

Hence S depends on q_s, q_s^* and t.

$$S = S(t, q_s, q_s^*) = S(t, q_s, \alpha_s)$$

where $q_s^* = \alpha_s$: at initial time every value will be zero so q_s^* = constant.

$$\Rightarrow \delta S = \frac{\partial S}{\partial t} \cancel{\delta t}^{\,0} + \frac{\partial S}{\partial q_s} \delta q_s + \frac{\partial S}{\partial \alpha_s} \delta \alpha_s$$

$$\Rightarrow \delta S = \frac{\partial S}{\partial q_s} \delta q_s + \frac{\partial S}{\partial \alpha_s} \delta \alpha_s \longrightarrow (4)$$

From (3) and (4), we have,

$$p_s \delta q_s - p_s^* \delta q_s^* = \frac{\partial S}{\partial q_s} \delta q_s + \frac{\partial S}{\partial \alpha_s} \delta \alpha_s$$

$$\Rightarrow p_s = \frac{\partial S}{\partial q_s} \quad \text{and} \quad -p_s^* = \frac{\partial S}{\partial \alpha_s} \text{ or } \beta_s = \frac{\partial S}{\partial \alpha_s}$$

$\because -p_s^* = \beta_s$

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Since $S = \int_{t_1}^{t_2} (p_s \dot{q}_s - H)\, dt$

So $\frac{dS}{dt} = p_s \dot{q}_s - H(t, q_s, p_s)$ ⟶ (5)

Also $S = S(t, q_s, \alpha_s)$.

So $\frac{dS}{dt} = \frac{\partial S}{\partial t}\frac{dt}{dt} + \frac{\partial S}{\partial q_s}\frac{dq_s}{dt} + \frac{\partial S}{\partial \alpha_s}\frac{d\alpha_s}{dt}$ (0)

$\Rightarrow \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_s}\dot{q}_s$ ⟶ (6)

Equating (5) and (6), we have,

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_s}\dot{q}_s = p_s \dot{q}_s - H(t, q_s, p_s).$$

$$\Rightarrow \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_s}\dot{q}_s = \frac{\partial S}{\partial q_s}\dot{q}_s - H\left(t, q_s, \frac{\partial S}{\partial q_s}\right)$$

$$\Rightarrow \frac{\partial S}{\partial t} + H\left(t, q_s, \frac{\partial S}{\partial q_s}\right) = 0$$

which is called Hamilton Jacobi equ

Cyclic / Ignorable Coordinates:- 2008, 2009, 2012, 2

A generalized coordinates q_i is called ignorable or cyclic if the Lagrangian L does not explicitly involve the gener coordinates q_i itself. However $\dot{q}_i$ is in Lagrangian function L.

In this case the Lagrange's equations o

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$$\frac{\partial}{\partial \beta_2}\left[\frac{\beta_2}{r^2}+2m\beta_1\right] dr + \int \frac{\partial}{\partial \beta_2}\sqrt{2mK\cos\theta - \beta_3^2\csc^2\theta - \beta_2}\, d\theta = \gamma_2$$

and $$\int \frac{\partial}{\partial \beta_3}\sqrt{2mK\cos\theta - \beta_3^2\csc^2\theta - \beta_2}\, d\theta = \gamma_3 - q \longrightarrow (21)$$

After solving (19), (20) and (21) we can find r, θ and q.

Hamilton Jacobi Theorem: 2014, 2018, 2008, :-

Let $S(t, q_s, \alpha_s)$ containing n arbitrary constan be a complete integral of the Hamilton Jacobi partial differential equations

$$\frac{\partial S}{\partial t} + H\left(t, q_s, \frac{\partial S}{\partial q_s}\right) = 0.$$

Then the solution of the Hamilton's equation

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} \quad ; s = 1, 2, \ldots, n.$$

is obtained by solving the algebraic equati

$$p_s = \frac{\partial S}{\partial q_s}, \quad \beta_s = \frac{\partial S}{\partial \alpha_s}$$ where β_s are new arbitrary constant.

Proof:

$H = \sum_{s=1} p_s \dot{q}_s - L$

Consider Hamilton's action integral with variable upper limit as t_2

$$S = \int_{t_1}^{t_2} L\, dt = \int_{t_1} (p_s \dot{q}_s - H)\, dt$$

where $H = p_s \dot{q}_s - L$ (E.S.C)

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Then $\delta S = \delta \int_{t_1}^{t_2} (p_s \dot{q}_s - H) dt$

$= \int_{t_1}^{t_2} \delta (p_s \dot{q}_s - H) dt$ where $H = H$

$= \int_{t_1}^{t_2} (\delta p_s \dot{q}_s + p_s \delta \dot{q}_s - \frac{\partial H}{\partial q_s} \delta q_s - \frac{\partial H}{\partial p_s}$

$- \frac{\partial H}{\partial t} \delta t^{\to 0}) dt$

$= \int_{t_1}^{t_2} (\delta p_s \dot{q}_s + p_s \delta \frac{dq_s}{dt} - \frac{\partial H}{\partial q_s} \delta q_s - \frac{\partial H}{\partial p_s} \delta p$

$\Rightarrow \delta S = \int_{t_1}^{t_2} (\delta p_s \dot{q}_s + p_s \frac{d}{dt} \delta q_s - \frac{\partial H}{\partial q_s} \delta q_s - \frac{\partial H}{\partial p_s} \delta p_s)$

Since the Hamilton's equations are

$\dot{p}_s = -\frac{\partial H}{\partial q_s}$ and $\dot{q}_s = \frac{\partial H}{\partial p_s}$

Therefore, equation (1) gives us

$\delta S = \int_{t_1}^{t_2} (\frac{\partial H}{\partial p_s} \delta p_s + p_s \frac{d}{dt} \delta q_s - \frac{\partial H}{\partial q_s} \delta q_s - \frac{\partial H}{\partial p_s} \delta p_s)$

$= \int_{t_1}^{t_2} (p_s \frac{d}{dt} \delta q_s) dt - \int_{t_1}^{t_2} (\frac{\partial H}{\partial q_s} \delta q_s) dt$

$\Rightarrow \delta S = I_1 - \int_{t_1}^{t_2} (\frac{\partial H}{\partial q_s} \delta q_s) dt \longrightarrow$

where $I_1 = \int_{t_1}^{t_2} (p_s \frac{d}{dt} \delta q_s) dt$

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(13) $\Rightarrow r^2\left(\frac{dS_1}{dr}\right)^2 = \beta_2 + 2m\beta_1 r^2$

$\Rightarrow \left(\frac{dS_1}{dr}\right)^2 = \frac{\beta_2}{r^2} + 2m\beta_1$

$\Rightarrow \frac{dS_1}{dr} = \sqrt{\frac{\beta_2}{r^2} + 2m\beta_1}$

$\Rightarrow S_1 = \int \sqrt{\frac{\beta_2}{r^2} + 2m\beta_1}\, dr$ ——— (15)

(14) $\Rightarrow \left(\frac{dS_2}{d\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{dS_3}{d\varphi}\right)^2 - 2mK\cos\theta = -\beta_2$

$\Rightarrow \sin^2\theta\left(\frac{dS_2}{d\theta}\right)^2 + \left(\frac{dS_3}{d\varphi}\right)^2 - 2mK\sin^2\theta\cos\theta = -\beta_2\sin^2\theta$

$\Rightarrow \sin^2\theta\left(\frac{dS_2}{d\theta}\right)^2 - 2mK\sin^2\theta\cos\theta + \beta_2\sin^2\theta = -\left(\frac{dS_3}{d\varphi}\right)^2$

$\Rightarrow -\sin^2\theta\left(\frac{dS_2}{d\theta}\right)^2 + 2mK\sin^2\theta\cos\theta - \beta_2\sin^2\theta = \left(\frac{dS_3}{d\varphi}\right)^2 \Rightarrow$

Since the L.H.S of (16) is function of θ only and the R.H.S is function of φ only, both sides are equal only if they are equal to some constant say β_3.

(16) $\Rightarrow \left(\frac{dS_3}{d\varphi}\right)^2 = \beta_3^2 \Rightarrow \frac{dS_3}{d\varphi} = \sqrt{\beta_3^2}$

$\Rightarrow S_3 = \int \beta_3\, d\varphi$ $\quad \because \sqrt{\beta_3^2} = \beta_3$

$\Rightarrow S_3 = \beta_3\varphi$ ——— (17)

and $-\sin^2\theta\left(\frac{dS_2}{d\theta}\right)^2 + 2mK\sin^2\theta\cos\theta - \beta_2\sin^2\theta = \beta_3^2$

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$$\Rightarrow \left(\frac{dS_2}{d\theta}\right)^2 = -\beta_3^2 \operatorname{cosec}^2\theta + 2mK\cos\theta - \beta_2$$

$$\Rightarrow \frac{dS_2}{d\theta} = \sqrt{2mK\cos\theta - \beta_3^2 \operatorname{cosec}^2\theta - \beta_2}$$

$$\Rightarrow S_2 = \int \sqrt{2mK\cos\theta - \beta_3^2 \operatorname{cosec}^2\theta - \beta_2}\, d\theta \longrightarrow (18)$$

Thus $S = \int\int \sqrt{\beta_2/i^2 + 2m\beta_1}\, di_2 + \int \sqrt{2mK\cos\theta - \beta_3^2 \operatorname{cosec}^2\theta - \beta_2}\, d\theta$

$+\beta_3 q - \beta_1 t$

By the Legendre equation $Q_s = \frac{\partial S}{\partial \beta_s}$

$$Q_1 = \frac{\partial S}{\partial \beta_1} \Rightarrow \gamma_1 = \frac{\partial S}{\partial \beta_1} \quad (\text{say}).$$

$$Q_2 = \frac{\partial S}{\partial \beta_2} \Rightarrow \gamma_2 = \frac{\partial S}{\partial \beta_2} \quad (\text{say})$$

$$Q_3 = \frac{\partial S}{\partial \beta_3} \Rightarrow \gamma_3 = \frac{\partial S}{\partial \beta_3} \quad (\text{say})$$

$$\Rightarrow \gamma_1 = \frac{\partial}{\partial \beta_1}\int \sqrt{\beta_2/i^2 + 2m\beta_1}\, di_2 - t$$

$$\gamma_2 = \partial/\partial\beta_2 \int \sqrt{\beta_2/i^2 + 2m\beta_1}\, di_2 + \frac{\partial}{\partial \beta_2}\int \sqrt{2mK\cos\theta - \beta_3^2 - \beta_2}$$

$$\gamma_3 = \partial/\partial\beta_3 \int \sqrt{2mK\cos\theta - \beta_3^2 \operatorname{cosec}^2\theta - \beta_2}\, d\theta + q$$

$$\Rightarrow \int \frac{\partial}{\partial \beta_1} \sqrt{\beta_2/i^2 + 2m\beta_1}\, di_2 = \gamma_1 + t \quad \text{——} (19)$$

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Solution.

The K.E in spherical coordinates (r, θ, φ) is

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2)$$

and $V = \frac{-K}{r^2}\cos\theta$ (given)

Note.
{See page 138-139 first 5 equati

The Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2\sin^2\theta} - \frac{K}{r^2}\cos\theta \quad \text{---(6)}$$

The Hamilton Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2\sin^2\theta} - \frac{K}{r^2}\cos\theta = 0.$$

$$\Rightarrow \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{2mr^2\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 - \frac{K}{r^2}\cos\theta = 0. \quad \text{---(7)}$$

Suppose $S = S_1(r) + S_2(\theta) + S_3(\varphi) + S_4(t)$ is solution of equation (7).

Then $\frac{\partial S}{\partial r} = \frac{dS_1}{dr}$; $\frac{\partial S}{\partial \theta} = \frac{dS_2}{d\theta}$

$\frac{\partial S}{\partial \varphi} = \frac{\partial S_3}{\partial \varphi}$; $\frac{\partial S}{\partial t} = \frac{dS_4}{dt}$

Thus equation (7) becomes.

$$\frac{dS_4}{dt} + \frac{1}{2m}\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS_2}{d\theta}\right)^2 + \frac{1}{2mr^2\sin^2\theta}\left(\frac{dS_3}{d\varphi}\right)^2 - \frac{K}{r^2}\cos\theta =$$

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$$\Rightarrow \frac{1}{2m}\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS_2}{d\theta}\right)^2 + \frac{1}{2mr^2\sin^2\theta}\left(\frac{dS_3}{d\varphi}\right)^2 - \frac{K\cos}{r^2}$$

$$= -\frac{dS_4}{dt} \quad \text{——— (8)}$$

Since the L.H.S of (8) is function of r, θ, φ and R.H.S is function of t only, both sid are equal if they are equal to some con say β_1.

$$(8) \Rightarrow \frac{1}{2m}\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS_2}{d\theta}\right)^2 + \frac{1}{2mr^2\sin^2\theta}\left(\frac{dS_3}{d\varphi}\right)^2 - \frac{K}{r^2}$$

$$\text{and } -\frac{dS_4}{dt} = \beta_1 \quad \text{——— (10)}$$

$$(10) \Rightarrow S_4 = -\beta_1 t \quad \text{——— (11)}$$

$$(9) \Rightarrow r^2\left(\frac{dS_1}{dr}\right)^2 + \left(\frac{dS_2}{d\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{dS_3}{d\varphi}\right)^2 - 2mK\cos\theta =$$

$$\Rightarrow r^2\left(\frac{dS_1}{dr}\right)^2 - 2m\beta_1 r^2 = -\left(\frac{dS_2}{d\theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{dS_3}{d\varphi}\right)^2$$

Since the L.H.S of (12) is function of only and R.H.S is function of θ and φ bothe sides are equal only if theyare equal t some constant say β_2.

$$\Rightarrow r^2\left(\frac{dS_1}{dr}\right)^2 - 2m\beta_1 r^2 = \beta_2 \quad \text{——— (13)}$$

$$\text{and } -\left(\frac{dS_2}{d\theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{dS_3}{d\varphi}\right)^2 + 2mK\cos\theta = \beta_2$$

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$$\Rightarrow p_i = \frac{\partial L}{\partial \dot{q}_i} = \beta_i \longrightarrow (2)$$

$\Rightarrow$ The momenta conjugate to cyclic coordinates are constant.

Note that

$$L = L(q_j, \dot{q}_i, \dot{q}_j, t) \; ; \; i = 1, 2, \cdots, m; \; j = m+1, m+2, \cdots, n$$

$$; S = 1, 2, \cdots, m, m+1, \cdots n.$$

The δ-variation of the Lagrangian is given by

$$\delta L = \sum_{j=m+1}^{n} \frac{\partial L}{\partial q_j} \delta q_j + \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{j=m+1}^{n} \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j + \frac{\partial L}{\partial t} \delta$$

$$\Rightarrow \delta L = \sum_{j=m+1}^{n} \frac{\partial L}{\partial q_j} \delta q_j + \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{j=m+1}^{n} \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \longrightarrow (3)$$

Also note that

$$\delta\left(\sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right) = \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=1}^{m} \delta\left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i$$

$$\Rightarrow \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i = \delta\left(\sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right) - \sum_{i=1}^{m} \delta\left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i$$

Thus equation (3) becomes

$$\delta L = \sum_{j=m+1}^{n} \frac{\partial L}{\partial q_j} \delta q_j + \delta\left(\sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right) - \sum_{i=1}^{m} \delta\left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i + \sum_{j=m+1}^{n} \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j$$

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$$\Rightarrow \delta\left(L - \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) = \sum_{j=m+1}^{n} \frac{\partial L}{\partial q_j}\delta q_j - \sum_{i=1}^{m} \delta\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_{j=m+1}^{n} \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j$$

using equation ② in above equation, we get

$$\delta\left(L - \sum_{i=1}^{m} \beta_i \dot{q}_i\right) = \sum_{j=m+1}^{n} \frac{\partial L}{\partial q_j}\delta q_j - \sum_{i=1}^{m} \delta\beta_i \dot{q}_i + \sum_{j=m+1}^{n} \frac{\partial}{\partial}$$

Let us introduce a new function called Routhian function as follows

$$R = L - \sum_{i=1}^{m} \beta_i \dot{q}_i$$

Then

$$\delta R = \sum_{j=m+1}^{n} \frac{\partial L}{\partial q_j}\delta q_j + \sum_{j=m+1}^{n} \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j - \sum_{i=1}^{m} \dot{q}_i\, \delta\beta_i \longrightarrow$$

Note that

$$R = R(q_j, \dot{q}_j, \beta_i, t).$$

The δ-variation of Routhian function given by

$$\delta R = \sum_{j=m+1}^{n} \frac{\partial R}{\partial q_j}\delta q_j + \sum_{j=m+1}^{n} \frac{\partial R}{\partial \dot{q}_j}\delta \dot{q}_j + \sum_{i=1}^{m} \frac{\partial R}{\partial \beta_i}\delta\beta_i + \frac{\partial R}{\partial t}\delta t \to 0$$

$$\Rightarrow \delta R = \sum_{j=m+1}^{n} \frac{\partial R}{\partial q_j}\delta q_j + \sum_{j=m+1}^{n} \frac{\partial R}{\partial \dot{q}_j}\delta \dot{q}_j + \sum_{i=1}^{m} \frac{\partial R}{\partial \beta_i}\delta\beta_i \longrightarrow$$

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Comparing (4) and (5), we get,

$\frac{\partial R}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}$, $\frac{\partial R}{\partial q_j} = \frac{\partial L}{\partial \dot{q}_j}$ and $\frac{\partial R}{\partial \dot{p}_i} = -\dot{q}_i$

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad ; \quad j = m+1, m+2, \ldots, n$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) - \frac{\partial R}{\partial q_j} = 0 \quad ; j = m+1, m+2, \ldots, n.$$

which is called Routhian equation of motion for non ignorable coordinates.

Example:

For a canonical pendulum the Lagrangian function is given by

$$L = \frac{1}{2} m (l^2\dot{\theta}^2 + l^2 \sin^2\theta \dot{\psi}^2) + mgl\cos\theta \quad \text{—— (1)}$$

Solution.

Note that $\theta, \dot{\theta}, \dot{\psi}$ are present whereas ψ is absent so ψ is cyclic coordinate.

The L.E.O.M. for ψ-coordinate is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \cancel{\frac{\partial L}{\partial \psi}}^{\,0} = 0 \quad \text{—— (2)} \quad \because L \neq L(\psi)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\psi}} = \text{Constant} = (\text{say}) \; P_\psi$$

or $$\frac{\partial L}{\partial \dot{\psi}} = P_\psi = P_\psi \quad \text{——} \quad (3)$$

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$$① \Rightarrow \frac{\partial L}{\partial \dot{\psi}} = m s^2 \sin^2\theta \dot{\psi} = p_\psi \text{ (say)}$$

$$\Rightarrow \dot{\psi} = \frac{p_\psi}{m s^2 \sin^2\theta} \quad \text{——} \quad ④$$

The Routhian function is given by

$$R = L - \sum_{i=1}^{m} p_i \dot{q}_i = L - p_\psi \dot{\psi}$$

$$\Rightarrow R = L - \frac{p_\psi^2}{m s^2 \sin^2\theta} \quad \text{——} \quad ⑤ \quad \text{:using ④}$$

$$④ \text{ in } ① \Rightarrow L = \frac{1}{2} m \left(s^2 \dot{\theta}^2 + s^2 \sin^2\theta \frac{p_\psi^2}{m^2 s^4 \sin^4\theta} \right) + mgs$$

$$\Rightarrow L = \frac{1}{2} m \left(s^2 \dot{\theta}^2 + \frac{p_\psi^2}{m^2 s^2 \sin^2\theta} \right) + mgs\cos\theta \quad \text{——}$$

$$⑥ \text{ in } ⑤ \Rightarrow R = \frac{1}{2} m s^2 \dot{\theta}^2 + \frac{1}{2} \frac{p_\psi^2}{m s^2 \sin^2\theta} + mgs\cos\theta - \frac{p_\psi^2}{ms^2}$$

$$\Rightarrow R = \frac{1}{2} m s^2 \dot{\theta}^2 - \frac{1}{2} \frac{p_\psi^2}{m s^2 \sin^2\theta} + mgs\cos\theta$$

$$\Rightarrow R = R(\theta, \dot{\theta}, p_\psi).$$

$$\frac{\partial R}{\partial \theta} = -\frac{1}{2} \frac{p_\psi^2}{ms^2} (-2 \operatorname{cosec}^2\theta \cot\theta) - mgs \sin$$

$$\Rightarrow \frac{\partial R}{\partial \theta} = \frac{p_\psi^2}{ms^2} \operatorname{cosec}^2\theta \cot\theta - mgs \sin\theta$$

$$\frac{\partial R}{\partial \dot{\theta}} = m s^2 \dot{\theta}$$

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The Routhian equation of motion is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt}(ml^2\dot{\theta}) - \frac{P_\psi^2}{ml^2}\frac{\cos\theta}{\sin^3\theta} + mgl\sin\theta = 0$$

$$\Rightarrow ml^2\ddot{\theta} - \frac{P_\psi^2}{ml^2}\frac{\cos\theta}{\sin^3\theta} + mgl\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} - \frac{P_\psi^2}{m^2l^4}\frac{\cos\theta}{\sin^3\theta} + \frac{g}{l}\sin\theta = 0$$

Quasi Coordinates. 2009, 2012, 2017, What are Quasi coordinates

In the analysis of rotational dynamics of rigid bodies, it is often convenient to use the angular velocity components $\omega_x, \omega_y, \omega_z$, directly as variables in the differential equations. These ω's given as dependent variables of Eules angles are

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$

$$\omega_y = \dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi$$

$$\omega_z = \dot{\phi} + \dot{\psi}\cos\theta$$

These equations are not integrable with respect to time so they cannot be taken as true velocities, because they cannot be expressed as functions of Eules's angles.

In general, if we express $d\pi_1 = \omega_x dt$; $d\pi_2 = \omega_y dt$; $d\pi_3 = \omega_z dt$ and $q_1 = \theta, q_2 = \phi$, $q_3 = \psi$, we find that $d\pi_s = a_{sk}\, dq_k$

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where s, k assume values 1, 2, 3.

Such type of coordinates are called Quas
Coordinates, which are the linear combinati
of generalized velocities.

Here $\bar{\pi}$'s are not expressible as functions
Euler's angles.

These coordinates are also called as Pseu
Coordinates.

Derive Lagrange's Equations of Motion in Quasi Coordinates (Boltzman Hamel Equations of m (Equations of motion involving derivatives o K.E. and Quasi Coordinates). **2009, 2012, 2017,** (aavib k notes sag krnahi)

The fundamental equation of motion of a for a rigid body is given by

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s\right]\delta q_s = 0 \quad \text{---------} \quad (1$$

where q's are the generalized coordinates.

Let $\bar{\pi}_\xi = \bar{\pi}_\xi(q_\beta)$ be the Quasi Coordinate

Then $\frac{d\bar{\pi}_\xi}{dt} = a_{s\xi}\frac{dq_\beta}{dt}$ $\quad \because d\bar{\pi}_\xi = a$

$\Rightarrow \dot{\bar{\pi}}_\xi = a_{s\xi}\dot{q}_\beta$

So that the generalized velocities $\dot{q}$'s ma
be expressed in terms of Quasi Coordin

$$\dot{q}_\beta = \frac{1}{a_{s\xi}}\dot{\bar{\pi}}_\xi$$

$\Rightarrow \dot{q}_\beta = b_{s\xi}\dot{\bar{\pi}}_\xi \quad \because b_{s\xi} = 1/a_{s\xi}$ (say).

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$$\Rightarrow \frac{dq_s}{dt} = b_{sr}\frac{d\pi_r}{dt}$$

$$\Rightarrow dq_s = b_{sr}\, d\pi_r$$

$$\Rightarrow \delta q_s = b_{sr}\,\delta\pi_r \qquad ; r = 1, 2, 3$$

Now equation (1) becomes

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s\right](b_{sr}\,\delta\pi_r) = 0$$

$$\Rightarrow \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) b_{sr} - \frac{\partial T}{\partial q_s} b_{sr} - Q_s b_{sr}\right]\delta\pi_r = 0 \qquad \text{---------- (2)}$$

Our next target is to express the K.E; T in terms of Quasi Coordinates, while deriving equation (1) T was function of q_s, $\dot{q}_s$ and time coordinate t We want to replace generalized velocities $\dot{q}_s$ by $\dot{\pi}_s$, so our target is

$T = T(t, q_s, \dot{q}_s)$ to be expressed as $T^*(t, q_s, \dot{\pi}_k$

$\Rightarrow T = T(t, q_s, \dot{q}_s) = T^*(t, q_s, \dot{\pi}_k)$.

We need to express $\frac{\partial T}{\partial \dot{q}_s}$ and hence its derivatives in terms of T^*.

Note that

$$\frac{\partial T}{\partial q_s} = \frac{\partial T^*}{\partial \dot{\pi}_k}\frac{\partial \dot{\pi}_k}{\partial q_s} \qquad \text{---------- (3)}$$

But $\dot{\pi}_k$ is linear combination of $\dot{q}_s$ as follows

$$\dot{\pi}_k = \frac{\partial \pi_k}{\partial q_s}\frac{dq_s}{dt} = \frac{\partial \pi_k}{\partial q_s}\dot{q}_s = a_{ks}\,\dot{q}_s$$

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$$\Rightarrow \dot{\bar{r}}_k = a_{ks}\dot{q}_s$$

Pal. Diff. w.r.t $\dot{q}_s$, we get

$$\frac{\partial \dot{\bar{r}}_k}{\partial \dot{q}_s} = a_{ks} \quad \text{---------- (4)}$$

(4) in (3) $\Rightarrow$ $\frac{\partial \bar{I}}{\partial \dot{q}_s} = \frac{\partial \bar{I}^*}{\partial \dot{\bar{r}}_k} a_{ks}$

Diff above relation w.r.t t, we have,

$$\frac{d}{dt}\left(\frac{\partial \bar{I}}{\partial \dot{q}_s}\right) = \frac{d}{dt}\left(\frac{\partial \bar{I}^*}{\partial \dot{\bar{r}}_k}\right) a_{ks} + \frac{\partial \bar{I}^*}{\partial \dot{\bar{r}}_k} \frac{d}{dt}(a_{ks})$$

we know that

$$a_{ks} = \frac{\partial \bar{r}_k}{\partial q_s} = a_{ks}(q_m)$$

$$\Rightarrow \frac{d}{dt}(a_{ks}) = \frac{\partial a_{ks}}{\partial q_m}\frac{dq_m}{dt}$$

$$\Rightarrow \frac{d}{dt}(a_{ks}) = \frac{\partial a_{ks}}{\partial q_m}\dot{q}_m \quad \text{---------- (6)}$$

(6) in (5) $\Rightarrow$ $\frac{d}{dt}\left(\frac{\partial \bar{I}}{\partial \dot{q}_s}\right) = a_{ks}\frac{d}{dt}\left(\frac{\partial \bar{I}^*}{\partial \dot{\bar{r}}_k}\right) + \frac{\partial \bar{I}^*}{\partial \dot{\bar{r}}_k}\frac{\partial a_{ks}}{\partial q_m}$

xing above equation by b_{sl} to get

$$\frac{d}{dt}\left(\frac{\partial \bar{I}}{\partial \dot{q}_s}\right) b_{sl} = a_{ks} b_{sl} \frac{d}{dt}\left(\frac{\partial \bar{I}^*}{\partial \dot{\bar{r}}_k}\right) + \frac{\partial \bar{I}^*}{\partial \dot{\bar{r}}_k}\frac{\partial a_{ks}}{\partial q_m}$$

Note that

$$a_{ks} b_{sl} = \left(\frac{\partial \bar{r}_k}{\partial q_s}\right)\left(\frac{\partial q_s}{\partial \bar{r}_l}\right) = \frac{\partial \bar{r}_k}{\partial \bar{r}_l}$$

$$\Rightarrow a_{ks} b_{sl} = \delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

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